Georgia Tech – Lorraine
Spring 2020
Differential Equations
Math 2552
30/1/2020

Last Name:
First Name:

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Quiz n^0 2 (20 minutes)

Show your work and justify your answers. Calculators, notes, cell phones, books are not allowed. Please do not use red or pink ink. Maximum: 20 points

Exercise 1 (3+3 points) .
Classify the following differential equations as separable, linear, exact, or none of these.
Do not attempt to solve the differential equation. Justify your answers.

1. $(y^2+1)+(y+2xy)\dfrac{dy}{dx}=0.$

- $y+2xy = y(1+2x)$. The DE is separable: for $1+2x \neq 0$, it can be written as

$$\frac{dy}{dx} = -\frac{(1+y^2)}{y}\,\frac{1}{1+2x} \quad \text{or} \quad -\frac{y}{1+y^2}\frac{dy}{dx} = \frac{1}{1+2x}$$

- It is not linear, for instance because of the term y^2 (or of $y\frac{dy}{dx}$)
- Set $M(x,y) = y^2+1$, $N(x,y) = y+2xy$. Then $\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x}$. So the DE is exact.

The fact that the DE is exact can also be obtained recalling that "separable" implies "exact"

2. $(x^2+y)+(1+2x)\dfrac{dy}{dx}=0.$

- The DE is not separable as x, y in x^2+y cannot be separated
- The DE is linear as of the form $a_0(x)\frac{dy}{dx} + a_1(x)y = g(x)$ where $a_0(x) = 1+2x$, $a_1(x) = 1$, $g(x) = -x^2$, [Not demanded: standard form is $\frac{dy}{dx} + \frac{1}{1+2x}y = -\frac{x^2}{1+2x}$; it is non-homogeneous]
- set $M(x,y) = x^2+y$, $N(x,y) = 1+2x$. Then $\frac{\partial M}{\partial y} = 1 \neq 2 = \frac{\partial N}{\partial x}$. So the DE is not exact.

REM: If one checks that the DE is not exact, one also obtains automatically that the DE cannot be separable.

Please turn $\longrightarrow$

Exercise 2 (1+4+3+4+2 points) . Consider the initial value problem corresponding to the linear differential equation

$$(x-1)\frac{dy}{dx}+\frac{2xy}{x+1}=1$$

with initial condition $y(0)=1$.

(a) Write the differential equation in standard form.

For $x \neq 1$, $\dfrac{dy}{dx}+\dfrac{2x}{(x-1)(x+1)}y=\dfrac{1}{x-1}$

(b) Determine the largest interval I where the solution of the initial value problem exists and is unique. *Justify your answer.*

The coefficients $p(x)=\dfrac{2x}{(x-1)(x+1)}$, $h(x)=\dfrac{1}{x-1}$ are defined and continuous on $(-\infty;-1)\cup(-1;1)\cup(1;+\infty)$. Since $0\in(-1;1)$, the largest interval I is $(-1;1)$

(c) Find an integrating factor for the differential equation.

$\dfrac{2x}{(x-1)(x+1)}=\dfrac{2x}{x^2-1}$ and $\displaystyle\int\frac{2x}{x^2-1}dx=\ln|x^2-1|+\text{constant}$

An antiderivative of $\dfrac{2x}{x^2-1}$ is therefore $A(x)=\ln|x^2-1|$. An integrating factor is $\mu(x)=e^{A(x)}=|x^2-1|$. Since $\mu(x)$ will multiply both sides of the DE (and the same $\pm$ sign will appear on both sides), we can also choose $\mu(x)=x^2-1$

(d) Find the general solution of the differential equation.

Multiply both sides of the DE by $\mu(x)$:

$$\underbrace{\mu(x)\frac{dy}{dx}+\frac{2x}{x^2-1}\mu(x)y}_{\frac{d}{dx}(\mu y)}=\frac{1}{x-1}\mu(x)=\frac{1}{x-1}(x^2-1)=x+1\text{, i.e.}$$

$\mu y=\int(x+1)dx=\frac{1}{2}x^2+x+C$. Thus $y=\dfrac{1}{x^2-1}\left(\frac{1}{2}x^2+x+C\right)$, where $C=$ constant

(e) Find the solution of the given initial value problem.

We determine C from the initial condition:

$1=y(0)=\dfrac{1}{-1}\left(\frac{1}{2}\cdot 0+0+C\right)$ donne $C=-1$. Replace into the general solution and get the solution of the IVP: $y=\dfrac{1}{x^2-1}\left(\frac{1}{2}x^2+x-1\right)$