Georgia Tech – Lorraine
Spring 2020
Differential Equations
Math 2552
4/2/2020

Last Name:
First Name:

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Quiz n^0 3 (make-up)

- *Please email your solution to* angela.pasquale@univ-lorraine.fr *or* angela.pasquale@georgiatech-metz.fr *by 12:15 pm (Atlanta time). Write* "Quiz 3" *in the subject.*
- *You can write your solution on the same pdf I sent you, for instance if you have a tablet or you can print it. If you need extra space, add as many pages as needed.*
  Otherwise, please write your solutions on blank paper. You do not need to copy the questions, just clearly mark the number of the exercise and separate the different exercises with a horizontal line.
- *Show your work and justify your answers. Please organize your work clearly, neatly, and legibly. Identify your answers.*
- *The solution of this quiz must be your own. Do not show, discuss or compare your solution with anybody else.*
- *During the time interval from the release time and submission deadline, I will be online. If you have questions about the quiz, you can send me email messages or Canvas messages.*
- *Maximum: 20 points.*

Exercise 1 (6 points) Transform the initial value problem $y'' - y' = e^t$ with initial conditions $y(0) = 1$ and $y'(0) = 2$, into an equivalent initial value problem for a first order system.
Write your answer in matrix form.

Set $\begin{cases} x_1 = y \\ x_2 = y' \end{cases}$, so $\begin{cases} x_1' = y' = x_2 \\ x_2' = y'' = y' + e^t = x_2 + e^t \end{cases}$, with initial conditions

$\begin{cases} x_1(0) = y(0) = 1 \\ x_2(0) = y'(0) = 2 \end{cases}$.. In matrix form, this is

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \end{pmatrix}$$, with initial condition $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Exercise 2 (7 points) .

Consider the homogeneous system of linear DE's: $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(x) \end{pmatrix}$ and $\mathbf{A}$ is a 3×3 matrix with real entries.

Suppose that the following are pairs of eigenvalues and corresponding eigenvectors of $\mathbf{A}$:

- $\lambda_1 = -1, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$,
- $\lambda_2 = 1, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$,
- $\lambda_3 = 2, \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

Write a fundamental set of solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
Explain why the solutions are linearly independent.

$\mathbf{x}_1(t) = e^{-t}\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$, $\mathbf{x}_2(t) = e^{t}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{x}_3(t) = e^{2t}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is a fundamental set of solutions. To prove that these solutions are linearly independent:

METHOD 1 : Compute their Wronskian $W[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3](t) = e^{-t}e^{t}e^{2t}\begin{vmatrix} 1 & -1 & 2 \\ 4 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix} =$

$= e^{2t}(-1 - 1 + 16 + \cancel{2} - \cancel{2} + 4) = e^{2t} \cdot 18 \neq 0$ pour tout t. They are hence lin. indep.

METHOD 2 (without computations) $\mathbf{A}$ is a 3×3 matrix with three distinct eigenvalues. So $\mathbf{A}$ is nondefective. The three solutions of the form $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$, where λ is one of the eigenvalues and $\mathbf{v}$ is a fixed eigenvector of $\mathbf{A}$ for the eigenvalue λ, are lin. indep. by Theorem 6.3.1

Please turn $\longrightarrow$

Exercise 3 (7 points) . Consider the homogeneous system of linear DE's: $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $\mathbf{A}$ is a 2×2 matrix with real entries.

Suppose that $\lambda_1 = 2 - i$ is a complex eigenvalue of $\mathbf{A}$ with corresponding eigenvector $\mathbf{v} = \begin{pmatrix} 2 \\ 3i \end{pmatrix}$.

Determine the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ in terms of real-valued functions.

The real-valued general solution is

$$\mathbf{x}(t) = C_1 \operatorname{Re} \mathbf{x}_1(t) + C_2 \operatorname{Im} \mathbf{x}_1(t)$$

where $\mathbf{x}_1(t) = e^{(2-i)t}\begin{pmatrix} 2 \\ 3i \end{pmatrix} = e^{2t} e^{-it}\begin{pmatrix} 2 \\ 3i \end{pmatrix} = e^{2t}(\cos(t) - i\sin(t))\begin{pmatrix} 2 \\ 3i \end{pmatrix}$

$$= e^{2t}\begin{pmatrix} 2\cos(t) - 2i\sin(t) \\ 3\sin(t) + 3i\cos(t) \end{pmatrix} = \underbrace{e^{2t}\begin{pmatrix} 2\cos(t) \\ 3\sin(t) \end{pmatrix}}_{\operatorname{Re}\mathbf{x}_1(t)} + i\underbrace{e^{2t}\begin{pmatrix} -2\sin(t) \\ 3\cos(t) \end{pmatrix}}_{\operatorname{Im}\mathbf{x}_1(t)}$$

Thus

$$\mathbf{x}(t) = e^{2t}\left[C_1 \begin{pmatrix} 2\cos t \\ 3\sin t \end{pmatrix} + C_2 \begin{pmatrix} -2\sin t \\ 3\cos t \end{pmatrix} \right], \quad C_1, C_2 \text{ constants}, \ t \in \mathbb{R}$$