Georgia Tech – Lorraine
Spring 2020
Differential Equations
Math 2552
3/29/2020

Last Name:	
First Name:	

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Quiz n^0 4

- Due Saturday, April 4, at noon (Atlanta time).
- The solution of the quiz must be your own. You may not show, discuss or compare your solutions with anyone else.
- Please return your quiz by email to:

  angela.pasquale@univ-lorraine.fr or angela.pasquale@georgiatech-metz.fr

  Write "Quiz 4" in your email's subject.
  Receipt will be acknowledged by email.
- Please check that your scanned solution is readable.
- Coverage of this quiz: Chapter 3 and Chapter 4, sections 4.1 to 4.3.
- Please do not use red or pink ink. If you are use a pencil, be sure that it is dark enough.
- Maximum: 20 points

In this quiz you will be asked to use the MIT Mathlet, Linear Phase Portraits: Matrix Entry, available at: https://mathlets.org/mathlets/linear-phase-portraits-matrix-entry/

Some indications about the MIT Mathlet "Linear Phase Portraits: Matrix Entry":

When the `[Companion Matrix]` option is selected, the first row entries of the displayed matrix are fixed to be 0 and 1. By deselecting the `[Companion Matrix]` option, you can choose all four entries a, b, c, d of the matrix. The values of a, b, c, d can be fixed between -4 et 4 using the corresponding sliders.
The point in the upper-left window gives the determinant and the trace of the displayed matrix.
If you select the `[eigenvalues]` option, the eigenvalues of the matrix become visible: their values are displayed and their location is plotted in the complex plane.
The big window on the upper-right corner of the screen shows the phase plane of the system (the coordinates are denoted by x and y instead of x_1 and x_2 as in the lectures). It displays the trajectories of a few solutions.
Placing the cursor on a point of the phase plane displays its (x, y)-coordinates below the bottom left corner of the phase plane. Clicking produces the trajectory passing through that point. You can clear all the trajectories using `[Clear]`, and return to the original set of trajectories by re-setting one of the sliders for the matrix entries.

Exercise 1 (4+3+3=10 points) Consider the system of linear DE's $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{pmatrix} 2 & -3 \\ -1 & 0 \end{pmatrix}$ and $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$.

REM: they can also be viewed in the MIT Mathlets. See at the end

(a) Determine its general solution.

Eigenvalues of A: $\det(A-\lambda I)=0 \Leftrightarrow \begin{vmatrix} 2-\lambda & -3 \\ -1 & -\lambda \end{vmatrix}=0 \Leftrightarrow -\lambda(2-\lambda)-3=0 \Leftrightarrow \lambda^2-2\lambda-3=0 \Leftrightarrow \lambda = 1\pm 2 \begin{cases} 3 \\ -1 \end{cases}$

Eigenvector for $\lambda_1=-1$; $(A+I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 3 & -3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow -x_1+x_2=0 \Leftrightarrow x_2=x_1$

choose e.g. $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Eigenvector for $\lambda_2=3$: $(A-3I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -1 & -3 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow x_1+3x_2=0 \Leftrightarrow x_1=-3x_2$

choose e.g. $v_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

The solutions $\mathbf{x}_1(t)=e^{-t}v_1$ and $\mathbf{x}_2(t)=e^{3t}v_2$ are linearly independent since $\lambda_1\neq\lambda_2$

The general solution of $\mathbf{x}'=A\mathbf{x}$ is therefore

$$\boxed{\mathbf{x}(t) = C_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{3t}\begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad , C_1, C_2 \text{ real constants}, t\in\mathbb{R}}$$

(b) Enter the matrix $\mathbf{A}$ into the MIT Mathlets application. A few trajectories are displayed in the phase portrait. A trajectory crosses the x-axis at $x = 2$. What is the solution having this as a trajectory assuming that this crossing occurs at $t = 0$?

We have to find the solution of $\mathbf{x}'=A\mathbf{x}$ with initial condition $\mathbf{x}(0)=\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \mathbf{x}(0) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} -3 \\ 1 \end{pmatrix} \Leftrightarrow \begin{cases} C_1-3C_2=2 \\ C_1+C_2=0 \end{cases} \Leftrightarrow \begin{cases} 4C_2=-2 \\ C_1=-C_2 \end{cases} \Leftrightarrow \begin{cases} C_1=1/2 \\ C_2=-1/2 \end{cases}$

Thus the solution is $\boxed{\mathbf{x}(t) = \frac{1}{2}e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}e^{3t}\begin{pmatrix} -3 \\ 1 \end{pmatrix}}$ or $\begin{cases} x(t) = \frac{1}{2}e^{-t} + \frac{3}{2}e^{3t} \\ y(t) = \frac{1}{2}e^{-t} - \frac{1}{2}e^{3t} \end{cases}$

(See Mathlet picture at the end)

(c) Write the equation of the solution $\mathbf{x}(t)$ whose trajectory is the half-line in the 3^{rd} quadrant (i.e. where $x < 0$, $y < 0$) and so that $\mathbf{x}(0) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$.

The solutions having trajectory on half lines are those of the form $C_1e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $C_2e^{3t}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$. The trajectory is in the 3^{rd} quadrant when the solution is $\mathbf{x}(t)=C_1e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with $C_1<0$. Requiring that $\mathbf{x}(0)=\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ fixes C_1, namely $\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \mathbf{x}(0) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Leftrightarrow C_1=-1$. The required solution is therefore

$$\boxed{\mathbf{x}(t) = -e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad , t\in\mathbb{R}.}$$

(The Mathlet picture is at the end)

Please turn ⟶

Exercise 2 (4+3+3 points) . Consider second-order linear differential equation $x'' - 4x' + 3x = 0$ where $x = x(t)$ is the unknown function.

(a) Find its general solution.

The characteristic equation $\lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1) = 0$ has two distinct real solutions $\lambda_1 = 1$, $\lambda_2 = 3$. The general solution is therefore

$$\boxed{x(t) = C_1 e^t + C_2 e^{3t}, \quad C_1, C_2 \text{ constants}, \ t \in \mathbb{R}}$$

PLEASE SEE REMARKS ON THIS QUESTION AT THE END OF THIS FILE

(b) Consider the associated dynamical system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ (i.e. the associated system of linear DE). Enter the matrix $\mathbf{A}$ into the MIT Mathlets application. A trajectory crosses the x-axis at $x = 2$. What is the solution of $x'' - 4x' + 3x = 0$ which corresponds to this trajectory if we assume that this crossing occurs at $t = 0$?

The associated system of 1st order DE is $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $A = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix}$, $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}$ $\underset{y(t)}{=}$

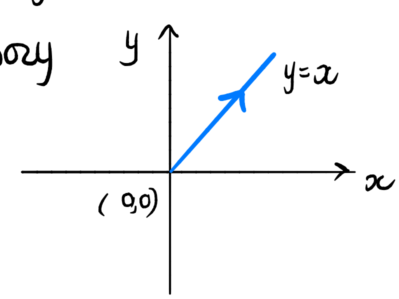
The general solution of the associated system is hence

$\mathbf{x}(t) = \begin{pmatrix} C_1 e^t + C_2 e^{3t} \\ C_1 e^t + 3C_2 e^{3t} \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. The solution $\mathbf{x}(t)$ crossing the x-axis at $x = 2$ when $t = 0$ is the initial condition $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, i.e. $\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} C_1 + C_2 \\ C_1 + 3C_2 \end{pmatrix}$

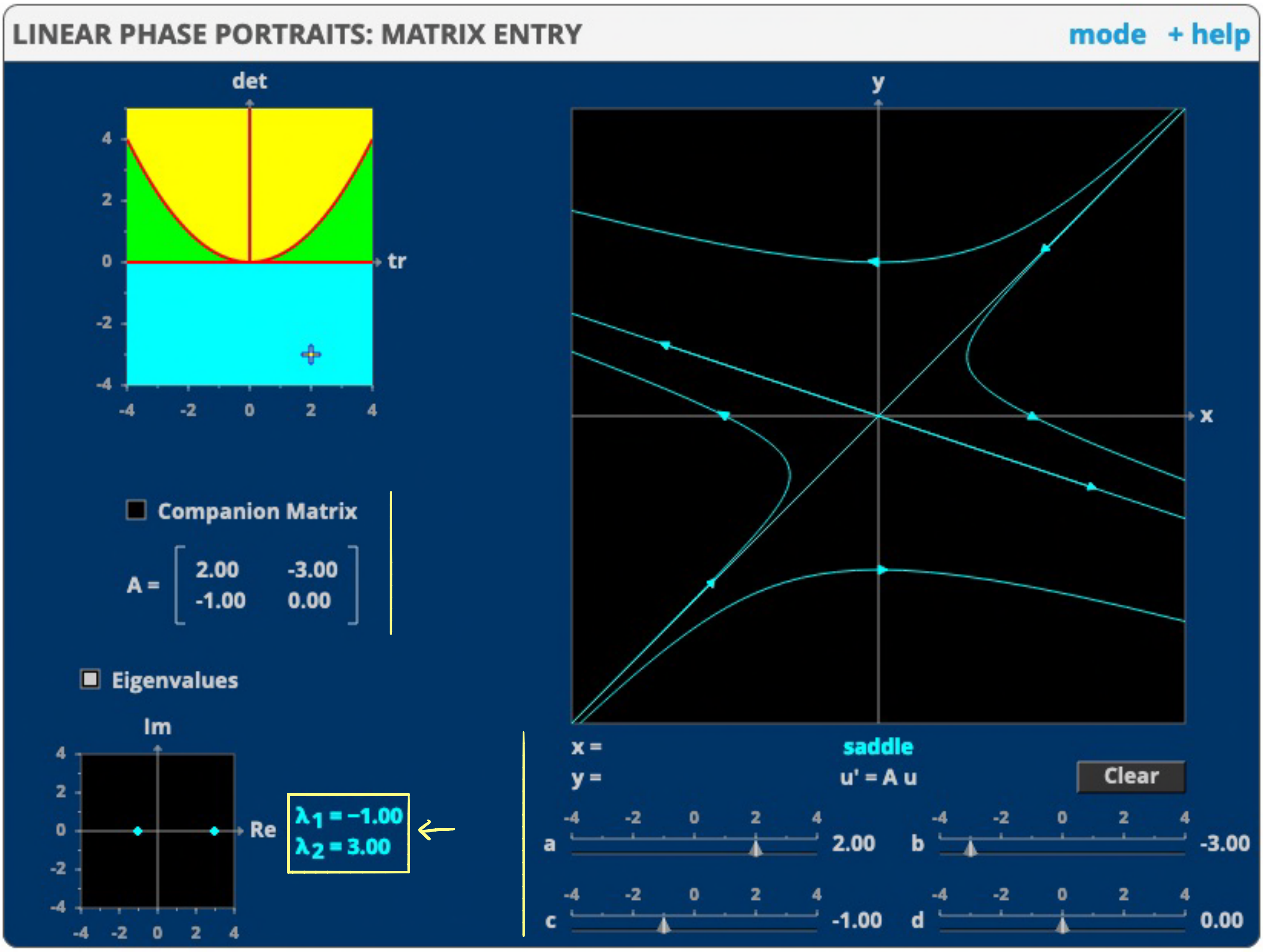
Hence $\begin{cases} C_1 + C_2 = 2 \\ C_1 + 3C_2 = 0 \end{cases}$, i.e. $\begin{cases} C_2 = -1 \\ C_1 = 3 \end{cases}$. The corresponding solution of $x'' - 4x' + 3x = 0$ is $\boxed{x(t) = 3e^t - e^{3t}}$.

(c) Sketch in the phase plane the trajectory corresponding to the solution $x(t) = e^t$ of $x'' - 4x' + 3x = 0$.

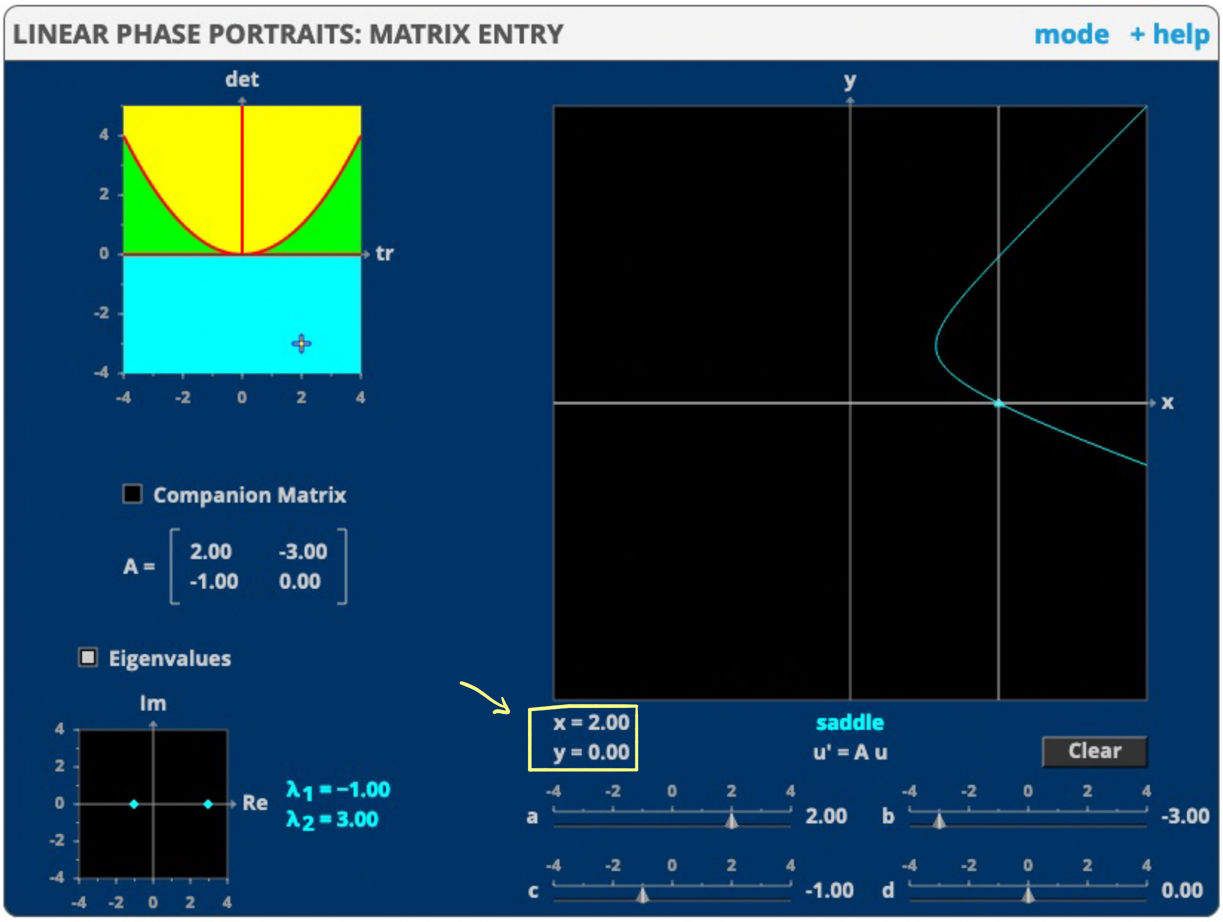
If $x(t) = e^t$, then $\underbrace{x'(t)}_{y(t)} = e^t$. Hence we need to sketch the trajectory of $\mathbf{x}(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix} = e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. If $\begin{cases} x(t) = e^t \\ y(t) = e^t \end{cases}$, then $\frac{y}{x} = 1$, i.e. the trajectory lies on the straight line $y = x$. Since $x(t) = e^t$ describes $(0, +\infty)$ as $t \in (-\infty, +\infty)$, the trajectory is the half-line from 0 (excluded) to ∞ on $y = x$ inside the 1st quadrant $(x>0, y>0)$; since $\lim_{t \to +\infty} e^t = +\infty$, the trajectory is directed away from the origin $(0,0)$



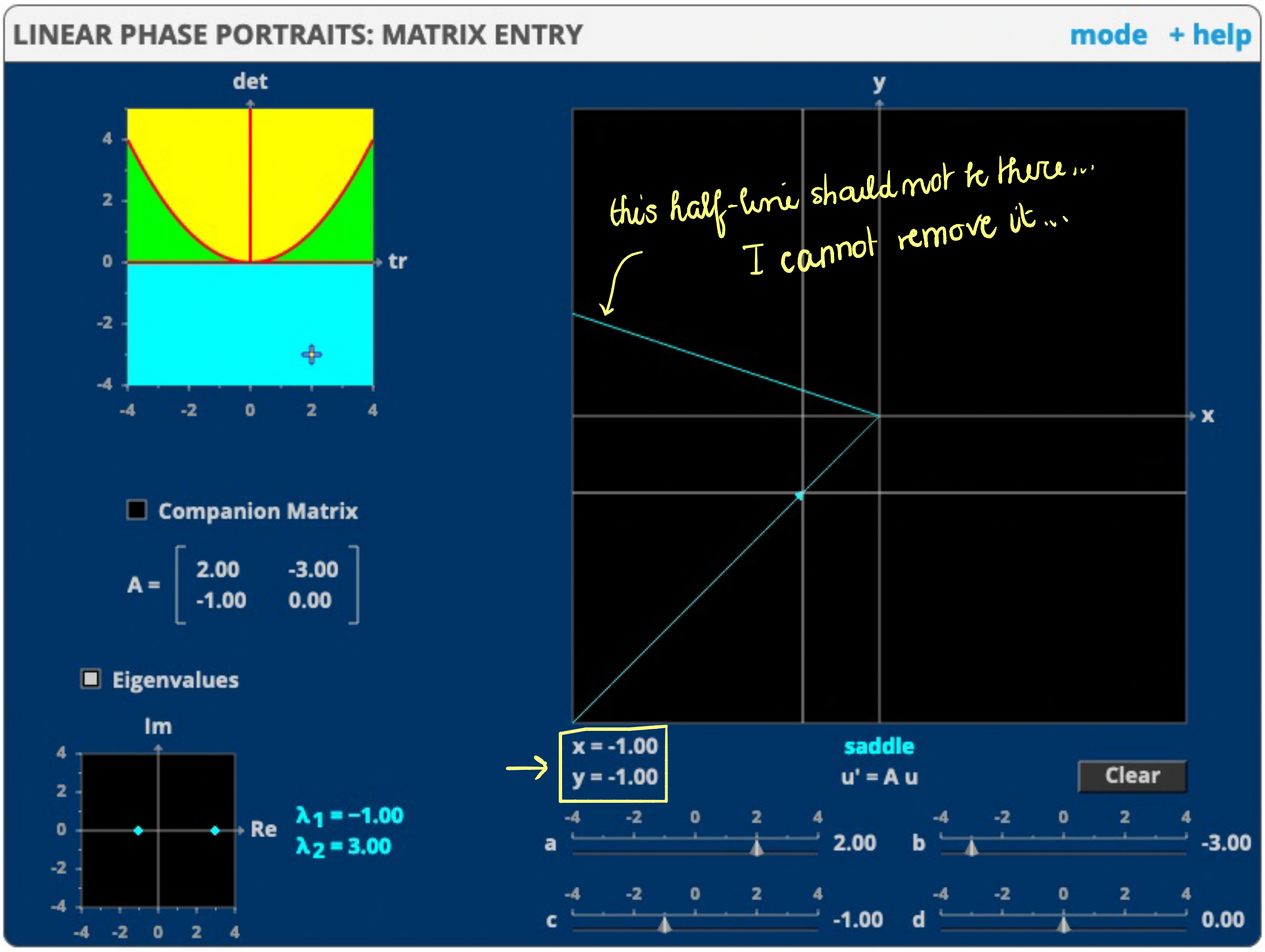
EXERCISE 1


LINEAR PHASE PORTRAITS: MATRIX ENTRY
mode + help
det
tr
Companion Matrix
A = [2.00 -3.00; -1.00 0.00]
Eigenvalues
Im
Re
λ1 = −1.00
λ2 = 3.00
y
x
x =
y =
saddle
u' = A u
Clear
a 2.00
b -3.00
c -1.00
d 0.00

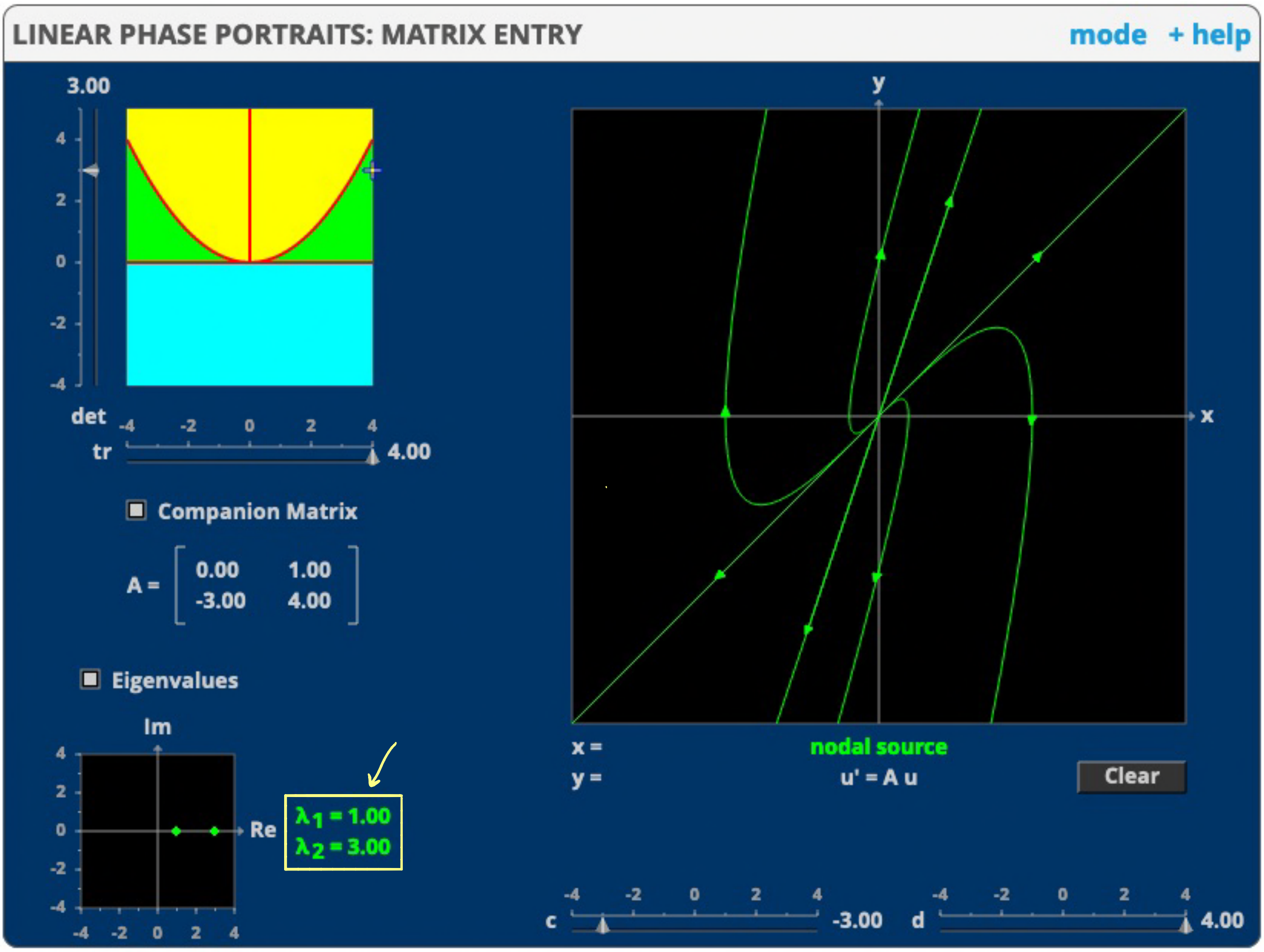
(b)


LINEAR PHASE PORTRAITS: MATRIX ENTRY
mode + help
det
tr
Companion Matrix
A = [2.00 -3.00; -1.00 0.00]
Eigenvalues
Im
Re
λ1 = −1.00
λ2 = 3.00
y
x
x = 2.00
y = 0.00
saddle
u' = A u
Clear
a 2.00
b -3.00
c -1.00
d 0.00

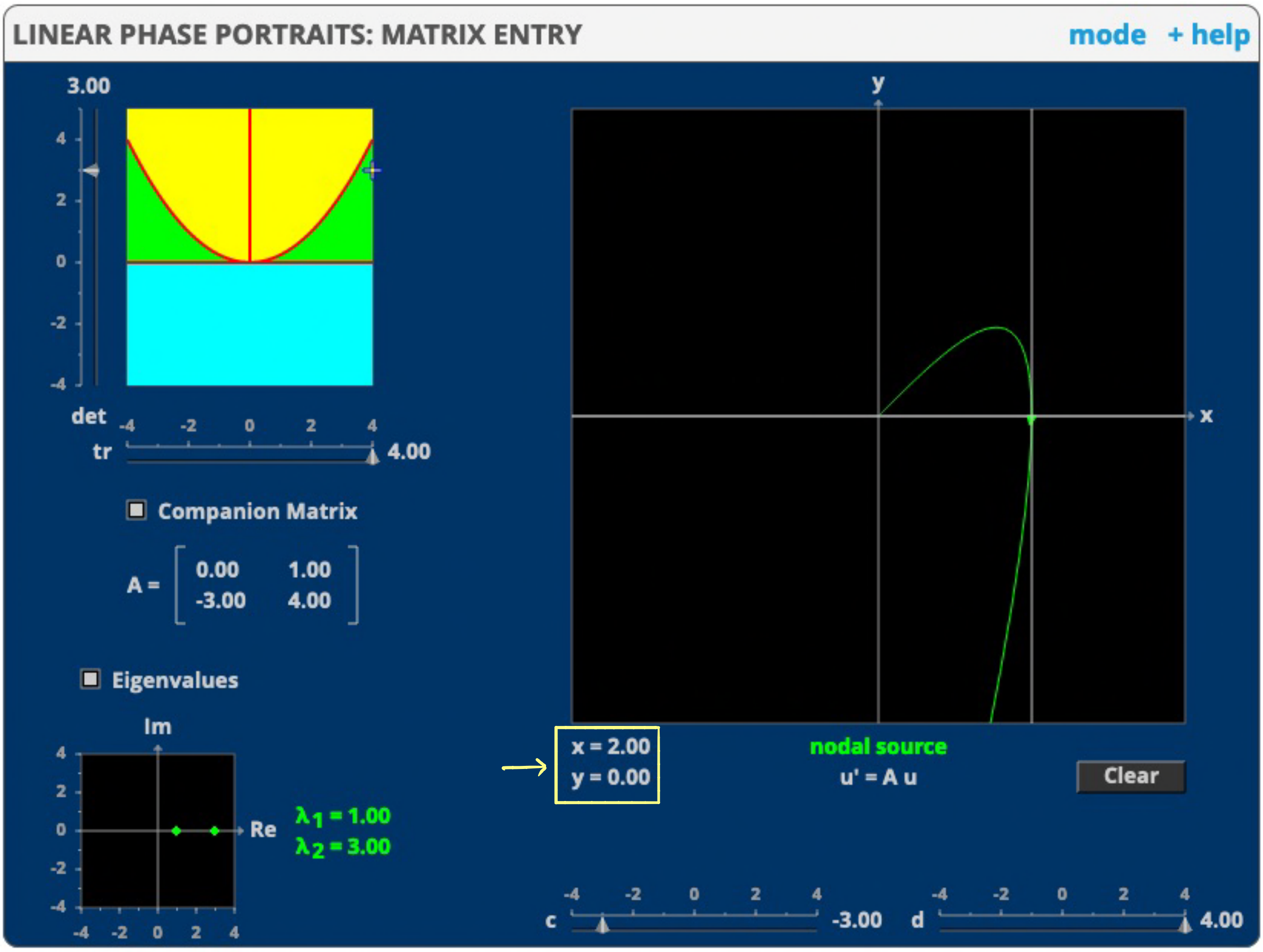
(c)


LINEAR PHASE PORTRAITS: MATRIX ENTRY
mode + help
det
tr
this half-line should not be there...
I cannot remove it...
x
y
Companion Matrix
A = [2.00 -3.00; -1.00 0.00]
Eigenvalues
Im
Re
λ1 = −1.00
λ2 = 3.00
x = -1.00
y = -1.00
saddle
u' = A u
Clear
a 2.00
b -3.00
c -1.00
d 0.00

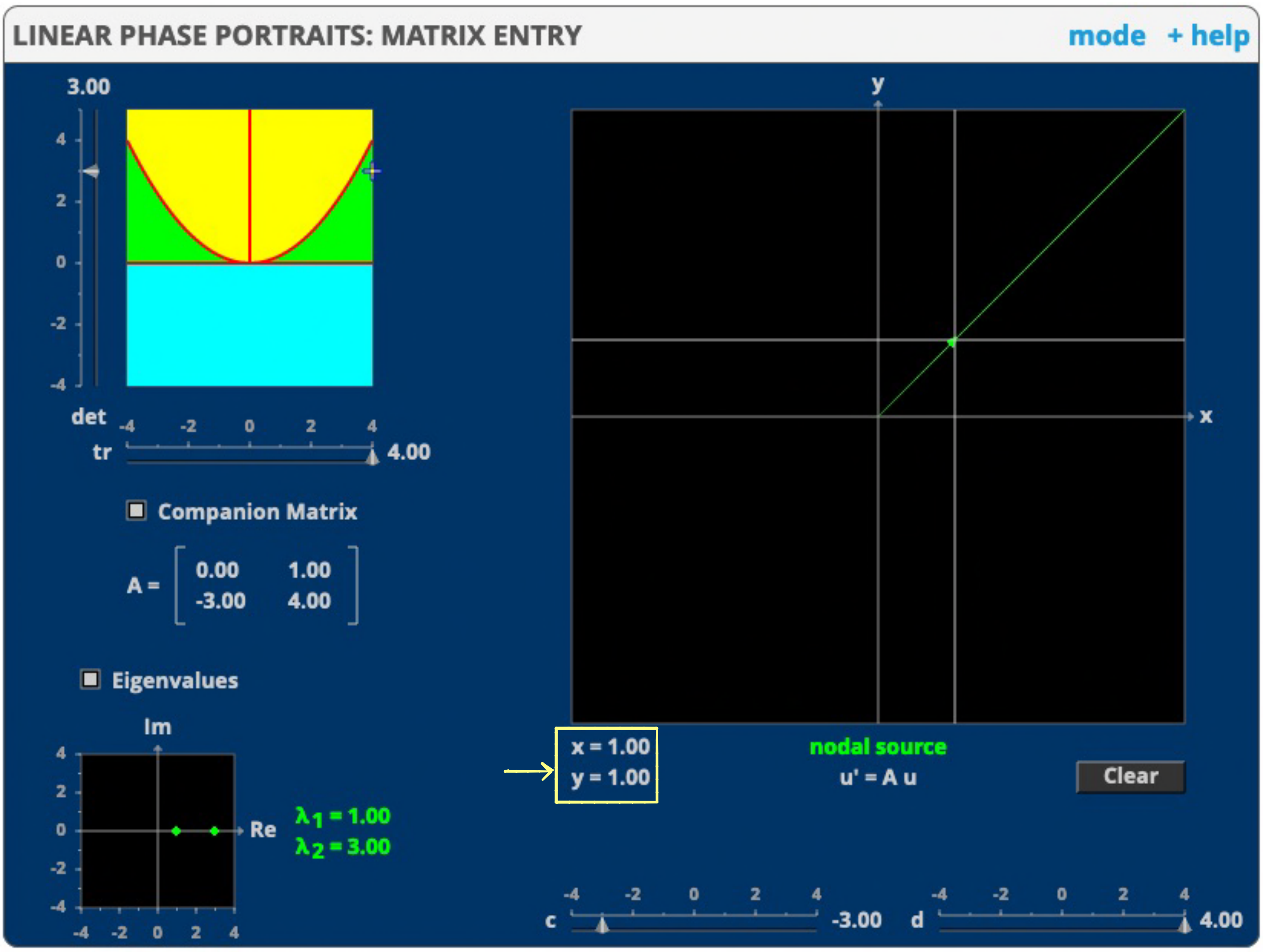
EXERCISE 2


LINEAR PHASE PORTRAITS: MATRIX ENTRY
mode + help
3.00
det
tr 4.00
y
x
Companion Matrix
A = [0.00 1.00; -3.00 4.00]
Eigenvalues
Im
Re
λ1 = 1.00
λ2 = 3.00
x =
y =
nodal source
u' = A u
Clear
c -3.00
d 4.00

(b)


LINEAR PHASE PORTRAITS: MATRIX ENTRY
mode
+ help
3.00
det
tr
4.00
Companion Matrix
A =
0.00 1.00
-3.00 4.00
Eigenvalues
Im
Re
λ1 = 1.00
λ2 = 3.00
y
x
x = 2.00
y = 0.00
nodal source
u' = A u
Clear
c
-3.00
d
4.00

(c)


LINEAR PHASE PORTRAITS: MATRIX ENTRY
mode
+ help
3.00
det
tr
4.00
Companion Matrix
A =
0.00 1.00
-3.00 4.00
Eigenvalues
Im
Re
λ1 = 1.00
λ2 = 3.00
y
x
x = 1.00
y = 1.00
nodal source
u' = A u
Clear
c
-3.00
d
4.00

SOME REMARKS ON THE SOLUTION OF EXERCISE 2 (a)

1. To find the general solution of a 2nd order homogeneous diff. equation with constant coefficients $ay'' + by' + cy = 0$ (where $a, b, c \in \mathbb{R}$), one does not need to pass to the associated system of 1st order DE's. The form of the general solution only depends on the roots of the characteristic equation $a\lambda^2 + b\lambda + c = 0$. See Theorem 4.3.2, p.5 of the slides of Section 4.3

2. One has to pass to the associated system of DE's to picture trajectories in the phase portrait. Namely: if $y = y(t)$ is a solution to $ay'' + by' + cy = 0$, then the corresponding solution of the associated system is $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ where $\begin{cases} x_1(t) = y(t) \\ x_2(t) = y'(t) \end{cases}$

<u>Example</u>: if $y(t) = e^t$ is a solution of $y'' - 4y' + 3y = 0$, then the corresponding solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix}$, is

$$\mathbf{x}(t) = \begin{pmatrix} e^t \\ (e^t)' \end{pmatrix} = \begin{pmatrix} e^t \\ e^t \end{pmatrix} = e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

<u>Notational remark</u>: here I am using the notation of the lectures. As remarked in the statement of Quiz 4, the MIT Mathlets employ a slightly different notation: the DE is $ax'' + bx' + cx = 0$, with unknown function $x = x(t)$. The variables in the phase portraits are (x, y). This means that the associated system is $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. If $x(t) = e^t$ is a solution of $x'' - 4x' + 3x = 0$, then the corresponding solution for the associated system is $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ with $y(t) = x'(t)$. So $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$. One has to be careful in distinguishing $x(t)$ and $\mathbf{x}(t)$ [not bold versus bold]

3. Back to the notation from the lectures:

- If λ is a real root of the characteristic equation of $ay''+by'+cy=0$, then (1) $y(t)=e^{\lambda t}$ is a solution of $ay''+by'+cy=0$ (Thm. 4.3.1, p.3 of slides)

  (2) λ is an eigenvalue of the matrix $\mathbf{A}=\begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}$ of the system associated with $ay''+by'+cy=0$ (p.2 of slides)

  (3) An eigenvector of $\mathbf{A}$ for the eigenvalue λ is $\begin{pmatrix} 1 \\ \lambda \end{pmatrix}$ (p.2 of slides)

In particular, we do not need to compute the eigenvectors in the usual way [= by solving $(\mathbf{A}-\lambda \mathbf{I})\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$] to determine the solution $\mathbf{x}(t)=e^{\lambda t}\begin{pmatrix} 1 \\ \lambda \end{pmatrix}$ of $\mathbf{x}'=\mathbf{A}\mathbf{x}$.

All of this is of course true just because $\mathbf{A}$ has the very special form $\mathbf{A}=\begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}$.

- If we have a solution $\mathbf{x}(t)=\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ for $\mathbf{x}'=\mathbf{A}\mathbf{x}$ with $\mathbf{A}=\begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}$,

  then $y(t)=x_1(t)$ is a solution of $ay''+by'+cy=0$ and $x_2(t)=y'(t)$ (pp. 2-3 of the slides)