Georgia Tech – Lorraine
Spring 2020
Differential Equations
Math 2552
4/15/2020

Last Name:
First Name:

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Quiz n^0 5

- *Please email your solution to* angela.pasquale@univ-lorraine.fr *or* angela.pasquale@georgiatech-metz.fr *by 12:15 pm (Atlanta time). Write* "Quiz 5" *in the subject.*
  Double check that your file is readable and complete.
- *You can write your solution on the same pdf I sent you, for instance if you have a tablet or you can print it. If you need extra space, add as many pages as needed.*
  Otherwise, please write your solutions on blank paper. You do not need to copy the questions, just clearly mark the number of the exercise and separate the different exercises with a horizontal line.
- *Show your work and justify your answers. Please organize your work clearly, neatly, and legibly. Identify your answers.*
- *The solution of this quiz must be your own. Do not show, discuss or compare your solution with anybody else.*
- *A table of Laplace transforms is at the end of this file. You can use the table on your textbook if you prefer.*
- *During the time interval from the release time and submission deadline, I will be online. If you have questions about the quiz, you can send me email messages.*
- *Maximum: 20 points.*

(0) Formulas 15. and 16. in Table 5.3.1 are not in the program this semester. You will not find them here or in the final exam.

SOME IMPORTANT REMARKS:

(1) In general, $\mathcal{L}\{fg\} \neq \mathcal{L}\{f\}\mathcal{L}\{g\}$

Ex: $f=g=1$ (the constant function). Then $fg=1$

$\mathcal{L}\{1\}(s) = \frac{1}{s}$ (for $s>0$). So $\mathcal{L}\{1\cdot 1\} = \mathcal{L}\{1\} = \frac{1}{s}$ whereas $\mathcal{L}\{1\}\mathcal{L}\{1\} = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}$ (for $s>0$)

In particular, to compute $\mathcal{L}\{u_c(t)f(t-c)\}$, you must use 13. in the table of Laplace transforms (Table 5.3.1). This is NOT equal to $\mathcal{L}\{u_c(t)\}\mathcal{L}\{f(t-c)\}$

(2) In general, $\mathcal{L}^{-1}\{FG\} \neq \mathcal{L}^{-1}\{F\}\mathcal{L}^{-1}\{G\}$.

Ex $F(s)=G(s)=\frac{1}{s}$ for $s>0$. Then $(FG)(s) = F(s)G(s) = \frac{1}{s^2}$

$\mathcal{L}^{-1}\{FG\} = \mathcal{L}^{-1}\{\frac{1}{s^2}\} = t$ (see 2. in Table 5.3.1)

whereas $\mathcal{L}^{-1}\{F\}\mathcal{L}^{-1}\{G\} = \mathcal{L}^{-1}\{\frac{1}{s}\}\mathcal{L}^{-1}\{\frac{1}{s}\} = 1\cdot 1 = 1$ (see 1. in Table 5.3.1)

In particular, to compute $\mathcal{L}^{-1}\{F\}$ where F is a product of the form $F(s) = \frac{1}{as+b}\cdot\frac{1}{cs+d}$

you must make a partial fraction decomposition to rewrite F as a linear combination

$F(s) = \frac{A}{as+b} + \frac{B}{cs+d}$ and hence use the linearity of $\mathcal{L}^{-1}$ to have $\mathcal{L}^{-1}\{F\} = A\mathcal{L}^{-1}\{\frac{1}{as+b}\} + B\mathcal{L}^{-1}\{\frac{1}{cs+d}\}$

(which can be computed from Table 5.3.1 as $\frac{1}{as+b} = \frac{1}{a(s+b/a)}$. Similar argument applies when $as+b$ is replaced e.g. by s^2+a.... A suitable partial fraction decomposition is needed.

Exercise 1 (4+6 points) .

(a) Write the following function using the unit step function

$$f(t) = \begin{cases} t^2 & \text{if } 0 \le t < \pi \\ \cos(2t) & \text{if } \pi \le t < 2\pi \\ 0 & \text{if } t \ge 2\pi \end{cases}$$

$$f(t) = t^2 u_{0\pi}(t) + \cos(2t)\, u_{\pi,2\pi}(t)$$
$$= t^2(u_0(t) - u_\pi(t)) + \cos(2t)(u_\pi(t) - u_{2\pi}(t))$$
$$= t^2 u_0(t) + (\cos(2t) - t^2)\, u_\pi(t) - \cos(2t)\, u_{2\pi}(t)$$
$$= t^2 \quad + (\cos(2t) - t^2)\, u_\pi(t) - \cos(2t)\, u_{2\pi}(t), \text{ for } t \geq 0$$

(b) Find the Laplace transform of the function f in (a).

We rewrite the expression for f found in (a) as

$$f(t) = t^2 + f_1(t-\pi)\, u_\pi(t) - f_2(t-2\pi)\, u_{2\pi}(t)$$

where $\quad f_1(t-\pi) = \cos(2t) - t^2$, i.e. $f_1(t) = \underbrace{\cos(2(t+\pi))}_{\cos(2t+2\pi) = \cos(2t)} - (t+\pi)^2$

and $\quad f_2(t-2\pi) = \cos(2t)$, i.e. $f_2(t) = \cos(2(t+2\pi)) = \cos(2t)$

Recall that $\mathcal{L}\{u_c(t) f(t-c)\}(s) = e^{-cs}\mathcal{L}\{f\}(s)$ for all s where $\mathcal{L}\{f\}(s)$ is defined

Hence:

$$\mathcal{L}\{f\}(s) = \mathcal{L}\{t^2\}(s) + \mathcal{L}\{u_\pi(t) f_1(t-\pi)\}(s) - \mathcal{L}\{u_{2\pi}(t) f_2(t-2\pi)\}(s)$$

$$= \frac{2}{s^3} + e^{-\pi s}\mathcal{L}\{f_1\}(s) - e^{-2\pi s}\mathcal{L}\{f_2\}(s) \qquad [\text{if } s > 0]$$

$$= \frac{2}{s^3} + e^{-\pi s}\left[\mathcal{L}\{\cos(2t)\}(s) - \mathcal{L}\{\overbrace{(t+\pi)^2}^{t^2+2\pi t+\pi^2}\}(s)\right] - e^{-2\pi s}\mathcal{L}\{\cos(2t)\}(s)$$

$$= \frac{2}{s^3} + \left(e^{-\pi s} - e^{-2\pi s}\right)\frac{s}{s^2+4} - e^{-\pi s}\left(\frac{2}{s^3} + \frac{2\pi}{s^2} + \frac{\pi^2}{s}\right)$$

Please turn ⟶

Exercise 2 (10 points) . Solve the following initial value problem using Laplace transforms.

$$y'' + 2y' + 5y = e^{-t}\sin t \qquad \text{with initial conditions } y(0) = y'(0) = 0$$

Apply the Laplace transform $\mathcal{L}$ to both sides of the DE. Since $\mathcal{L}$ is linear, we obtain

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\sin t\} \qquad (*)$$

Set $Y = \mathcal{L}\{y\}$. The properties of the Laplace transform and the initial conditions give:

$$\mathcal{L}\{y''\}(s) = s^2\mathcal{L}\{y\}(s) - s\,y(0) - y'(0) = s^2 Y(s)$$
$$\mathcal{L}\{y'\}(s) = s\,\mathcal{L}\{y\}(s) - y(0) = s\,Y(s)$$

Moreover,

$$\mathcal{L}\{e^{-t}\sin t\}(s) = \mathcal{L}\{\sin t\}(s+1) \quad (\text{for } s > -1 \text{ ; see Thm 5.2.1 or n}^\circ 14 \text{ in the Tables of Laplace transforms } 5.8.1)$$
$$= \frac{1}{(s+1)^2+1} \quad (\text{Tables, n}^\circ 5)$$

Thus (*) becomes:

$$s^2Y(s) + 2sY(s) + 5Y(s) = \frac{1}{(s+1)^2+1}$$

$$(s^2+2s+5)\,Y(s) = \frac{1}{(s+1)^2+1}$$

$$Y(s) = \frac{1}{s^2+2s+5}\,\frac{1}{(s+1)^2+1} = \frac{1}{(s+1)^2+4}\,\frac{1}{(s+1)^2+1}$$

Both polynomials at the denominator are functions of $u = (s+1)^2$

$$\frac{A}{u+4} + \frac{B}{u+1} = \frac{(A+B)u + (A+4B)}{(u+1)(u+4)}, \text{ i.e. } \begin{cases} A+B=0 \\ A+4B=1 \end{cases} \rightarrow B = \frac{1}{3},\ A = -\frac{1}{3}$$

Hence $Y(s) = -\frac{1}{3}\,\frac{1}{(s+1)^2+4} + \frac{1}{3}\,\frac{1}{(s+1)^2+1}$. Thus:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}(t) + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$
$$= -\frac{1}{6}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\Big|_{s\to s+1}\right\}(t) + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\Big|_{s\to s+1}\right\}(t)$$
$$= -\frac{1}{6}e^{-t}\sin(2t) + \frac{1}{3}e^{-t}\sin t$$

Please turn for the tables of Laplace transforms ⟶

Elementary Laplace transforms.

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1.	1	$\frac{1}{s}, \quad s > 0$	Sec. 5.1; Ex. 4
2.	e^{at}	$\frac{1}{s-a}, \quad s > a$	Sec. 5.1; Ex. 5
3.	$t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$	Sec. 5.2; Cor. 5.2.5
4.	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 5.1; Prob. 37
5.	$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$	Sec. 5.1; Ex. 7
6.	$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$	Sec. 5.1; Prob. 22
7.	$\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 5.1; Prob. 19
8.	$\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 5.1; Prob. 18
9.	$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 5.1; Prob. 23
10.	$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 5.1; Prob. 24
11.	$t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 5.2; Prob. 8
12.	$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$	Sec. 5.5; Eq. (4)
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 5.5; Eq. (6)
14.	$e^{ct}f(t)$	$F(s-c)$	Sec. 5.2; Thm. 5.2.1
~~15.~~	$\int_0^t f(t-\tau)g(\tau)\,d\tau$ NOT DONE IN THE CLASS	$F(s)G(s)$	Sec. 5.6; Thm. 5.8.3
~~16.~~	$\delta(t-c)$ THIS SEMESTER	e^{-cs}	Sec. 5.7; Eq. (14)
17.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 5.2; Cor. 5.2.3
18.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$	Sec. 5.2; Thm. 5.2.4