RECITATION2 SPRING 2020

Tuesday, March 31, 2020 11:00 AM (Atlanta time)

4.1 Definitions and Examples

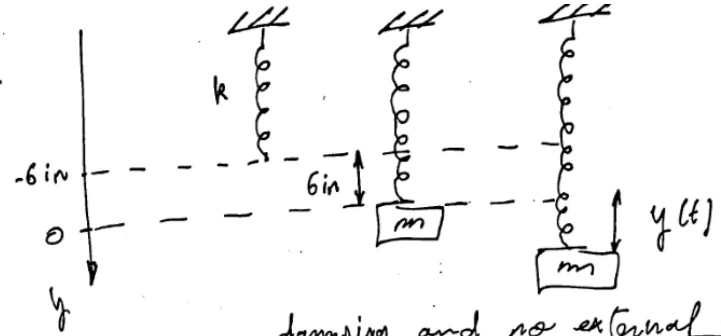
For each spring-mass system or electric circuit in Problems 10 through 17, write down the appropriate initial value problem based on the physical description.

10. A mass weighing 2 lb stretches a spring 6 in. The mass is pulled down an additional 3 in. and then released. Assume there is no damping.

11. A mass of 100 g stretches a spring 20 cm. The mass is set in motion from its equilibrium position with a downward velocity of 5 cm/s. Assume there is no damping.

15. A mass weighing 16 lb stretches a spring 3 in. The mass is attached to a viscous damper with a damping constant of 2 lb·s/ft. The mass is set in motion from its equilibrium position with a downward velocity of 3 in./s.

Pb10:



Here we have no damping and no external force so the differential equation of the system is

$$\boxed{my'' + ky = 0}$$

and k spring constant satisfying $W = mg = kL$

with $W = 2\,lb$; $L = 6\,in = \frac{1}{2}\,ft$

so $k = \frac{W}{L} = \frac{2\,lb}{\frac{1}{2}\,ft} = 4\,lb/ft$

and $m = \frac{W}{g} = \frac{2\,lb}{32\,ft/s^2} = \frac{1}{16}\,\frac{lb}{ft}\,s^2$

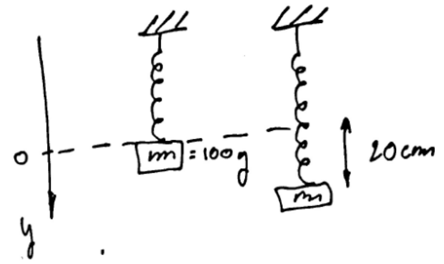
and the mass is pulled down by 3 in, so

$y(0) = 3\,in = \frac{1}{4}\,ft$

Pb 11: Here also we have no damping and no external force but we are in the metric system:

$\boxed{m y''(t) + k y(t) = 0}$

with: $\boxed{m = 0{,}1 \text{ kg}}$



so $W = kL = mg$

$\Rightarrow \quad k = \dfrac{mg}{L} = \dfrac{0{,}1 \text{ kg} \times 9{,}8 \text{ N/kg}}{0{,}2 \text{ m}}$

$\boxed{k = 4{,}9 \text{ N/m}}$

and $\underline{y(0) = 0}$

$\underline{y'(0) = 0{,}05 \text{ m/sec}}$ (positive because applied downward)

Pb 15: Here we have damping with a damping coefficient is $\gamma = 2$ lb-sec/ft

so the equation of motion is:

$\boxed{m y'' + \gamma y' + k y = 0}$

Since $W = kL = mg$

so $k = \dfrac{W}{L} = \dfrac{16 \text{ lb}}{3 \text{ in}} = \dfrac{16 \text{ lb}}{\left(\frac{1}{4}\right) \text{ ft}} = \boxed{64 \text{ lb/ft}}$

and $m = \dfrac{W}{g} = \dfrac{16 \text{ lb}}{32 \text{ ft/s}^2} = \boxed{\dfrac{1}{2} \dfrac{\text{lb s}^2}{\text{ft}}}$

so $m y'' + \gamma y' + k y = 0$

$\frac{1}{2} y'' + 2 y' + 64 y = 0$

$\boxed{y'' + 4 y' + 128 y = 0}$

with $\boxed{\begin{array}{l} y(0) = 0 \text{ ft} \\ y'(0) = \frac{1}{4} \text{ ft/sec} \end{array}}$

4.2 Theory of Second Order Linear Homogeneous Equations

PROBLEMS

In each of Problems 1 through 8, determine the longest interval in which the given initial value problem is certain to have a unique twice differentiable solution. Do not attempt to find the solution.

7. $(1-x^2)y'' - 2xy' + (\alpha(\alpha+1) + \mu^2/(1-x^2))y = 0,$ $y(0) = y_0, \quad y'(0) = y_1$

15. Verify that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are two solutions of the differential equation $t^2y'' - 2y = 0$ for $t > 0$. Then show that $c_1t^2 + c_2t^{-1}$ is also a solution of this equation for any c_1 and c_2.

7. Write the initial value problem as

$$y'' - \frac{2x}{1-x^2}y' + (\frac{\alpha(\alpha+1)}{1-x^2} + \frac{\mu^2}{(1-x^2)^2})y = 0.$$

Since the coefficient functions are continuous for all x such that $x \neq -1, 1$ and $x_0 = 0$, the initial value problem is guaranteed to have a unique solution for $-1 < x < 1$.

15. If $y_1 = t^2$, then $y_1'' = 2$. Therefore, $t^2y_1'' - 2y_1 = t^2(2) - 2t^2 = 0$. If $y_2 = t^{-1}$, then $y_2' = 2t^{-3}$. Therefore, $t^2y_2'' - 2y_2 = t^2(2t^{-3}) - 2t^{-1} = 0$. Since the equation is linear, the function $y_3 = c_1t^2 + c_2t^{-1}$ will also be a solution.

4.3 Linear Homogeneous Equations with Constant Coefficients

PROBLEMS

In each of Problems 1 through 26:

(a) Find the general solution in terms of real functions.

(b) From the roots of the characteristic equation, determine whether each critical point of the corresponding dynamical system is asymptotically stable, stable, or unstable, and classify it as to type.

(c) Use the general solution obtained in part (a) to find a two-parameter family of trajectories $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} = y\mathbf{i} + y'\mathbf{j}$ of the corresponding dynamical system. Then sketch by hand, or use a computer, to draw a phase portrait, including any straight-line orbits, from this family of trajectories.

1. $y'' + 2y' - 3y = 0$

3. $y'' - 4y' + 4y = 0$

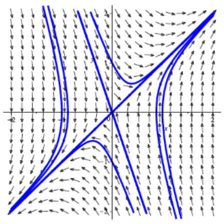
1.(a) The characteristic equation is given by $\lambda^2 + 2\lambda - 3 = 0$. Therefore, the two distinct roots are $\lambda = -3, 1$. Thus, the general solution is given by

$$y(t) = c_1 e^t + c_2 e^{-3t}.$$

(b) The critical point $(0, 0)$ is a saddle point, therefore, unstable.

(c) For y above, we see that $y' = c_1 e^t - 3c_2 e^{-3t}$. Therefore, we can rewrite our solution as the two parameter family

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} c_1 e^t + c_2 e^{-3t} \\ c_1 e^t - 3c_2 e^{-3t} \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-3t}.$$



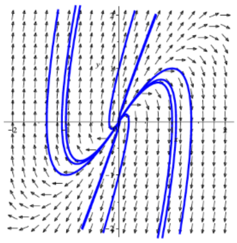
3.(a) The characteristic equation is given by $\lambda^2 - 4\lambda + 4 = 0$. Therefore, we have one repeated root $\lambda = 2$. Therefore, the general solution is given by

$$y(t) = c_1 e^{2t} + c_2 t e^{2t}.$$

(b) The critical point $(0, 0)$ is an unstable improper node.

(c) For y above, we see that $y' = 2c_1 e^{2t} + c_2(1 + 2t)e^{2t}$. Therefore, we can rewrite our solution as the two parameter family

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} + c_2 t e^{2t} \\ 2c_1 e^{2t} + c_2(1 + 2t)e^{2t} \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} t \\ 1 + 2t \end{pmatrix} e^{2t}.$$



In each of Problems 27 through 43, solve the given initial value problem. Sketch the graph of its solution and describe its behavior for increasing t.

27. $y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 1$

28. $y'' + 16y = 0, \quad y(0) = 0, \quad y'(0) = 1$

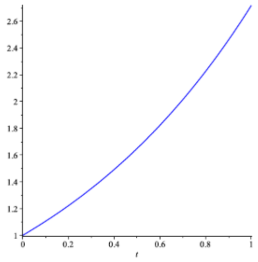
27. The characteristic equation is given by $\lambda^2 + \lambda - 2 = 0$. Therefore, the two distinct roots are $\lambda = -2, 1$. Therefore, the general solution is given by

$$y(t) = c_1 e^{-2t} + c_2 e^t.$$

Therefore,

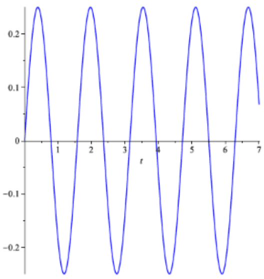
$$y'(t) = -2c_1 e^{-2t} + c_2 e^t.$$

Now using the initial conditions, we need $c_1 + c_2 = 1$ and $-2c_1 + c_2 = 1$. The solution of this system of equations is $c_1 = 0$ and $c_2 = 1$. Therefore, the specific solution is $y(t) = e^t$.



The solution $y \to \infty$ as $t \to \infty$.

28. The characteristic equation is $\lambda^2 + 16 = 0$, which has roots $\lambda = \pm 4i$. Therefore, the general solution is $y(t) = c_1 \cos 4t + c_2 \sin 4t$. The derivative of y is $y'(t) = -4c_1 \sin 4t + 4c_2 \cos 4t$. Using the initial conditions, we have $c_1 = 0$ and $4c_2 = 1$. Therefore, the solution is $y(t) = \frac{1}{4} \sin 4t$.



The solution will continue to oscillate with the same amplitude as $t \to \infty$.

4.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

PROBLEMS

In each of Problems 1 through 16, find the general solution of the given differential equation:

1. $y'' - 2y' - 3y = 3e^{2t}$

11. $2y'' + 3y' + y = t^2 + 3\sin t$

1. The characteristic equation for the homogeneous problem is $\lambda^2 - 2\lambda - 3 = 0$, which has roots $\lambda = 3, -1$. Therefore, the solution of the homogeneous problem is $y_h(t) = c_1e^{3t} + c_2e^{-t}$. To find a solution of the nonhomogeneous problem, we look for a solution of the form $y_p(t) = Ae^{2t}$. Substituting a function of this form into the differential equation, we have

$$4Ae^{2t} - 4Ae^{2t} - 3Ae^{2t} = 3e^{2t}.$$

Therefore, we need $-3A = 3$, or $A = -1$. Therefore, the general solution of the nonhomogeneous problem is

$$y(t) = c_1e^{3t} + c_2e^{-t} - e^{2t}.$$

More details:

We look for a solution of the form $y_p(t) = Ae^{2t}$

Substitution in Diff Eq: $4A\cancel{e^{2t}} - 4A\cancel{e^{2t}} - 3A\cancel{e^{2t}} = 3\cancel{e^{2t}}$

$\Rightarrow \boxed{A = -1}$

$\Rightarrow y(t) = y_h(t) + y_p(t)$

$\boxed{y(t) = c_1e^{3t} + c_2e^{-t} - e^{2t}}$

11. The characteristic equation for the homogeneous problem is $2\lambda^2 + 3\lambda + 1 = 0$, which has roots $\lambda = -1, -1/2$. Therefore, the solution of the homogeneous problem is $y_h(t) = c_1e^{-t} + c_2e^{-t/2}$. To find a solution of the nonhomogeneous problem, we will first look for a solution of the form $Y_1(t) = A + Bt + Ct^2$ to account for the nonhomogeneous term t^2. Substituting a function of this form into the differential equation, and equating like terms, we have $A + 3B + 4C = 0$, $B + 6C = 0$ and $C = 1$. The solution of these equations is $A = 14$, $B = -6$, $C = 1$. Therefore, $Y_1 = 14 - 6t + t^2$. Next, we look for a solution of the nonhomogeneous problem of the form $Y_2(t) = D\cos t + E\sin t$. Substituting this function into the differential equation and equating like terms, we find that $D = -9/10$ and $E = -3/10$. Therefore, the general solution of the nonhomogeneous problem is

$$y(t) = c_1e^{-t} + c_2e^{-t/2} + 14 - 6t + t^2 - \frac{9}{10}\cos t - \frac{3}{10}\sin t.$$

More details:

We look for a solution $Y_1(t) = A + Bt + Ct^2$

so $Y_1'(t) = B + 2Ct$

$Y_1''(t) = 2C$

$\Rightarrow$ we substitute in equation below:

$$2y'' + 3y' + y = t^2$$

so $2 \times 2C + 3(B + 2Ct) + (A + Bt + Ct^2) = t^2$

so $4C + 3B + 6Ct + A + Bt + Ct^2 = t^2$

Equating terms $\Rightarrow \begin{cases} A + 3B + 4C = 0 \\ B + 6C = 0 \\ C = 1 \end{cases}$

$\Rightarrow C = 1 \ ; \ B = -6 \ ; \ A = 14$

$\Rightarrow \boxed{Y_1(t) = 14 - 6t + t^2}$

Then, we look for a second solution $Y_2(t) = D\cos(t) + E\sin(t)$

$\Rightarrow Y_2'(t) = -D\sin(t) + E\cos(t)$

$Y_2''(t) = -D\cos(t) - E\sin(t)$

and we substitute in equation below:

$$2y'' + 3y' + y = 3\sin(t)$$

so $2Y_2''(t) + 3Y_2'(t) + Y_2(t) = 3\sin(t)$

$\Rightarrow (-E - 3D)\sin(t) + (-D + 3E)\cos(t) = 3\sin(t)$

$\Rightarrow$ by Equating terms : $\begin{cases} 3 = (-E - 3D) \\ 0 = -D + 3E \end{cases}$

$\Rightarrow \begin{cases} D = 3E \\ 3 = -10E \end{cases} \Rightarrow \boxed{E = \frac{-3}{10} \ \& \ D = \frac{-9}{10}}$

Comment: For t*e(t) and t*e(-t) we use Integration by parts

4.7 Variation of Parameters

In each of Problems 2 through 5, use the method of variation of parameters to find a particular solution using the given fundamental set of solutions $\{\mathbf{x}_1, \mathbf{x}_2\}$.

3. $\mathbf{x}' = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^{-t} \\ t \end{pmatrix}$,

$$\mathbf{x}_1 = e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{x}_2 = e^{t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

10. $y'' - 5y' + 6y = 2e^t$

3. Let $\mathbf{X}(t)$ be the fundamental matrix

$$\mathbf{X}(t) = \begin{pmatrix} e^{-t} & -e^{t} \\ 0 & e^{t} \end{pmatrix}.$$

Then

$$\mathbf{X}^{-1}(t) = \begin{pmatrix} e^{t} & e^{t} \\ 0 & e^{-t} \end{pmatrix}.$$

Therefore,

$$\mathbf{X}^{-1}(t)\mathbf{g}(t) = \begin{pmatrix} e^{t} & e^{t} \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} e^{-t} \\ t \end{pmatrix} = \begin{pmatrix} 1 + te^{t} \\ te^{-t} \end{pmatrix}.$$

Integrating by parts, we have

$$\int \mathbf{X}^{-1}(t)\mathbf{g}(t)\,dt = \int \begin{pmatrix} 1 + te^{t} \\ te^{-t} \end{pmatrix} dt = \begin{pmatrix} t + te^{t} - e^{t} \\ -te^{-t} - e^{-t} \end{pmatrix}.$$

Thus

$$\mathbf{x}_p(t) = \mathbf{X}(t)\int \mathbf{X}^{-1}(t)g(t)\,dt = \begin{pmatrix} e^{-t} & -e^{t} \\ 0 & e^{t} \end{pmatrix}\begin{pmatrix} t + te^{t} - e^{t} \\ -te^{-t} - e^{-t} \end{pmatrix} = \begin{pmatrix} te^{-t} + 2t \\ -t - 1 \end{pmatrix}.$$

THEOREM 4.7.1

Assume that the entries of the matrices $\mathbf{P}(t)$ and $\mathbf{g}(t)$ in Eq. (2) are continuous on an open interval I and that $\mathbf{x}_1$ and $\mathbf{x}_2$ are a fundamental set of solutions of the homogeneous equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ corresponding to the nonhomogeneous equation (1)

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t).$$

Then a particular solution of Eq. (1) is

$$\boxed{\mathbf{x}_p(t) = \mathbf{X}(t)\int \mathbf{X}^{-1}(t)\mathbf{g}(t)\,dt,} \qquad (15)$$

where the fundamental matrix $\mathbf{X}(t)$ is defined by Eq. (5). Moreover the general solution of Eq. (1) is

$$\boxed{\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \mathbf{x}_p(t).} \qquad (16)$$

10. The solution of the homogeneous equation is $y_h(t) = c_1e^{2t} + c_2e^{3t}$. The functions $y_1(t) = e^{2t}$ and $y_2(t) = e^{3t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{5t}$. Using the method of variation of parameters, a particular solution is given by $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ where

$$u_1(t) = -\int \frac{e^{3t}(2e^t)}{W(t)}\,dt = 2e^{-t} \quad \text{and} \quad u_2(t) = \int \frac{e^{2t}(2e^t)}{W(t)}\,dt = -e^{-2t}.$$

Therefore, the particular solution is $Y(t) = 2e^t - e^t = e^t$.

THEOREM 4.7.2

If the functions p, q, and g are continuous on an open interval I, and if the functions y_1 and y_2 form a fundamental set of solutions of the homogeneous equation $y''+p(t)y'+q(t)y=0$ corresponding to the nonhomogeneous equation (19),

$$y'' + p(t)y' + q(t)y = g(t),$$

then a particular solution of Eq. (19) is

$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W[y_1, y_2](t)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W[y_1, y_2](t)}\,dt. \tag{27}$$

The general solution is

$$y = c_1y_1(t) + c_2y_2(t) + Y(t), \tag{28}$$

as prescribed by Theorem 4.5.2.

4.4 Mechanical and Electrical Vibrations

PROBLEMS

In each of Problems 1 through 4, determine ω_0, R, and δ so as to write the given expression in the form $y = R\cos(\omega_0 t - \delta)$.

1. $y = 3\cos 2t + 3\sin 2t$

7. A mass weighing 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in., and then set in motion with a downward velocity of 2 ft/s, and if there is no damping, find the position y of the mass at any time t. Determine the frequency, period, amplitude, and phase of the motion.

13. A certain vibrating system satisfies the equation $y'' + \gamma y' + y = 0$. Find the value of the damping coefficient γ for which the quasi-period of the damped motion is 50% greater than the period of the corresponding undamped motion.

1. $R\cos\delta = 3$ and $R\sin\delta = 3$ implies $R = \sqrt{18} = 3\sqrt{2}$ and $\delta = \arctan(3/3) = \pi/4$. Therefore,

$$y = 3\sqrt{2}\cos(2t - \frac{\pi}{4}).$$

More details:

$$[R\cos(\delta)]^2 + [R\sin(\delta)]^2 = 3^2+3^2 = 9+9 = 18$$

$$R^2\underbrace{(\cos^2(\delta)+\sin^2(\delta))}_{=1} = 18$$

$$\Rightarrow \boxed{R^2 = 18} \Rightarrow \boxed{R = \sqrt{18} = 3\sqrt{2}}$$

and $$\frac{\cancel{R}\sin(\delta)}{\cancel{R}\cos(\delta)} = \frac{3}{3}$$

$$\Rightarrow \tan(\delta) = 1 \Rightarrow \boxed{\delta = \arctan(1) = \frac{\pi}{4}}$$

7. The spring constant is $k = 3/(1/4) = 12$ lb/ft. The mass is $3/32$ lb-s^2/ft. Therefore, the equation of motion is

$$\frac{3}{32}y'' + 12y = 0,$$

which can be simplified to $y'' + 128y = 0$. The initial conditions are $y(0) = -1/12$ ft, $y'(0) = 2$ ft/sec. The general solution is $y(t) = A\cos(8\sqrt{2}t) + B\sin(8\sqrt{2}t)$. Considering the initial conditions, we arrive at the solution

$$y(t) = -\frac{1}{12}\cos(8\sqrt{2}t) + \frac{\sqrt{2}}{8}\sin(8\sqrt{2}t).$$

The frequency $\omega_0 = 8\sqrt{2}$ rad/sec. The period is $T = \pi\sqrt{2}/8$ seconds. The amplitude is $R = \sqrt{(1/12)^2 + (\sqrt{2}/8)^2} = \sqrt{11/2}/12$ ft. The phase is $\delta = \pi - \arctan(3/\sqrt{2})$.

More details:

From initial conditions $\begin{cases} y(0) = -\frac{1}{12} \text{ ft} \\ y'(0) = 2 \text{ ft/sec} \end{cases}$

$\Rightarrow$ $y(t) = A\cos(8\sqrt{2}t) + B\sin(8\sqrt{2}t)$

$\boxed{y(0) = A + 0 = A = -\frac{1}{12}}$

& $y'(t) = -8\sqrt{2}A\sin(8\sqrt{2}t) + 8\sqrt{2}B\cos(8\sqrt{\ldots}$

$y'(0) = -8\sqrt{2}A \times 0 + 8\sqrt{2}B$

$y'(0) = 8\sqrt{2}B = 2$

$\Rightarrow$ $\boxed{B = \frac{\sqrt{2}}{8}}$

13. The frequency of the undamped motion is $\omega_0 = 1$. Therefore, the period of the undamped motion is $T = 2\pi$. The quasi-frequency of the damped motion is $\mu = \frac{1}{2}\sqrt{4-\gamma^2}$. Therefore, the period of the damped motion is $T_d = 4\pi/\sqrt{4-\gamma^2}$. We want to find γ such that $T_d = 1.5T$. That is, we want γ to satisfy $4\pi/\sqrt{4-\gamma^2} = 3\pi$. Solving this equation, we have $\gamma = 2\sqrt{5}/3$.