5.1 Definition of the Laplace Transform

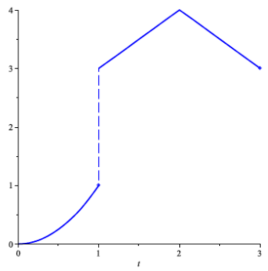
In each of Problems 1 through 4, sketch the graph of the given function. In each case, determine whether f is continuous, piecewise continuous, or neither on the interval $0 \le t \le 3$.

1. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 2+t, & 1 < t \le 2 \\ 6-t, & 2 < t \le 3 \end{cases}$

2. $f(t) = \begin{cases} \sin(\pi t), & 0 \le t \le 1 \\ (t-1)^{-1}, & 1 < t \le 2 \\ 1, & 2 < t \le 3 \end{cases}$

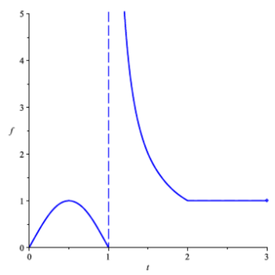
7. $e^{5t} \sin 3t$

1.



The function is piecewise continuous.

2.



The function is neither continuous nor piecewise continuous.

7. Since $|f(t)| = |e^{5t}\sin(3t)| \le e^{5t}$, the function is of exponential order. We can take $K = 1$, $a = 5$ and $M = 0$.

5.2 Properties of the Laplace Transform

In each of Problems 1 through 10, find the Laplace transform of the given function. Assume that a and b are real numbers and n is a positive integer.

1. $f(t) = e^{-2t} \sin 3t$ **2.** $f(t) = e^{3t} \cos 2t$

11. **(a)** Let $F(s) = \mathcal{L}\{f(t)\}$, where $f(t)$ is piecewise continuous and of exponential order on $[0, \infty)$. Show that

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s). \qquad \text{(i)}$$

Hint: Let $g_1(t) = \int_0^t f(t_1)\,dt_1$ and note that $g_1'(t) = f(t)$. Then use Theorem 5.2.2.

1. By Example 7 in Section 5.1, for $f(t) = \sin 3t$, $F(s) = \frac{3}{s^2+9}$. Therefore, by Theorem 5.2.1,

$$\mathcal{L}(e^{-2t}\sin 3t) = F(s+2) = \frac{3}{(s+2)^2+9}.$$

2. First, we compute the Laplace transform of $f(t) = \cos 2t$. For $f(t) = \cos 2t$,

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st}\cos 2t\,dt = \frac{1}{2}\int_0^\infty e^{-st}(e^{2it} + e^{-2it})\,dt \\ &= \frac{1}{2}\int_0^\infty (e^{-(s-2i)t} + e^{-(s+2i)t})\,dt = \lim_{b\to\infty}\frac{1}{2}\left[\frac{e^{-(s-2i)t}}{-(s-2i)} + \frac{e^{-(s+2i)t}}{-(s+2i)}\bigg|_0^b\right] \\ &= \frac{1}{2}\left[\frac{1}{s-2i} + \frac{1}{s+2i}\right] = \frac{2}{s^2+4}. \end{aligned}$$

Therefore, by Theorem 5.2.1,

$$\mathcal{L}(e^{3t}\cos 2t) = F(s-3) = \frac{s-3}{(s-3)^2+4}.$$

11.(a) Let $g_1(t) = \int_0^t f(t_1)\,dt_1$. Therefore, $g_1'(t) = f(t)$ which implies $\mathcal{L}(g_1'(t)) = \mathcal{L}(f(t)) = F(s)$. Also, by Theorem 5.2.2, $\mathcal{L}(g_1'(t)) = s\mathcal{L}(g_1(t)) - g_1(0) = s\mathcal{L}(g_1(t))$. Therefore, $s\mathcal{L}(g_1(t)) = \mathcal{L}(g_1'(t)) = F(s)$, so

$$\mathcal{L}(g_1(t)) = \frac{F(s)}{s}.$$

That is,

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}F(s).$$

5.3 The Inverse Laplace Transform

In each of Problems 9 through 24, use the linearity of $\mathcal{L}^{-1}$, partial fraction expansions, and Table 5.3.1 to find the inverse Laplace transform of the given function:

9. $\frac{30}{s^2+25}$ **10.** $\frac{4}{(s-3)^3}$

17. $\frac{1-2s}{s^2+4s+5}$

9. Writing the function as

$$\frac{30}{s^2+25} = 6\frac{5}{s^2+5^2},$$

we see that $\mathcal{L}^{-1}(Y(s)) = 6\sin 5t$.

10. Writing the function as

$$\frac{4}{(s-3)^3} = 2\frac{2}{(s-3)^3},$$

we see that $\mathcal{L}^{-1}(Y(s)) = 2t^2e^{3t}$.

17. Completing the square in the denominator, we have

$$\frac{1-2s}{s^2+4s+5} = \frac{5-2(s+2)}{(s+2)^2+1}.$$

Therefore, we see that $\mathcal{L}^{-1}(Y(s)) = 5e^{-2t}\sin t - 2e^{-2t}\cos t$.