5.4 Solving Differential Equations with Laplace Transforms

PROBLEMS

In each of Problems 1 through 13, use the Laplace transform to solve the given initial value problem:

1. $y'' - 4y' - 12y = 0; \quad y(0) = 8, \quad y'(0) = 0$

7. $y'' + \omega^2 y = \cos 2t, \quad \omega^2 \neq 4;$
$y(0) = 1, \quad y'(0) = 0$

In each of Problems 14 through 19, use the Laplace transform to solve the given initial value problem:

19. $\mathbf{y}' = \begin{pmatrix} 2 & -64 \\ 1 & -14 \end{pmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

1. Applying the Laplace transform to the differential equation, we get $\mathcal{L}(y'') - 4\mathcal{L}(y') - 12\mathcal{L}(y) = 0$, which is $[s^2\mathcal{L}(y) - sy(0) - y'(0)] - 4[s\mathcal{L}(y) - y(0)] - 12\mathcal{L}(y) = 0$. Using the initial conditions, this gives $[s^2\mathcal{L}(y) - 8s] - 4[s\mathcal{L}(y) - 8] - 12\mathcal{L}(y) = 0$, which turns into $[s^2 - 4s - 12]\mathcal{L}(y) = 8s - 32$, thus (using partial fractions)

$$\mathcal{L}(y) = \frac{8s - 32}{s^2 - 4s - 12} = \frac{2}{s-6} + \frac{6}{s+2},$$

which implies that

$$y(t) = 2e^{6t} + 6e^{-2t}.$$

7. Applying the Laplace transform to the differential equation, we get $\mathcal{L}(y'') + \omega^2\mathcal{L}(y) = \mathcal{L}(\cos 2t)$, which is

$$[s^2\mathcal{L}(y) - sy(0) - y'(0)] + \omega^2\mathcal{L}(y) = \frac{s}{s^2+4}.$$

Using the initial conditions, this gives

$$[s^2\mathcal{L}(y) - s] + \omega^2\mathcal{L}(y) = \frac{s}{s^2+4},$$

which turns into

$$[s^2 + \omega^2]\mathcal{L}(y) = s + \frac{s}{s^2+4},$$

and then

$$\mathcal{L}(y) = \frac{s}{s^2+\omega^2} + \frac{s}{(s^2+\omega^2)(s^2+4)}.$$

Using partial fractions in the second term, we rewrite $\mathcal{L}(y)$ as

$$\mathcal{L}(y) = \frac{s}{s^2+\omega^2} + \frac{1}{4-\omega^2}\left[\frac{s}{s^2+\omega^2} - \frac{s}{s^2+4}\right],$$

which implies

$$y = \cos\omega t + \frac{1}{4-\omega^2}[\cos\omega t - \cos 2t] = \frac{1}{4-\omega^2}[(5-\omega^2)\cos\omega t - \cos 2t].$$

19. The system can be written as

$$\begin{aligned} y_1' &= 2y_1 - 64y_2 \\ y_2' &= y_1 - 14y_2 \end{aligned}$$

with the initial conditions $y_1(0) = 0$ and $y_2(0) = 1$. Applying the Laplace transform to each equation, we have

$$\begin{aligned} \mathcal{L}(y_1') &= 2\mathcal{L}(y_1) - 64\mathcal{L}(y_2) \\ \mathcal{L}(y_2') &= \mathcal{L}(y_1) - 14\mathcal{L}(y_2), \end{aligned}$$

which implies

$$\begin{aligned} s\mathcal{L}(y_1) - y_1(0) &= 2\mathcal{L}(y_1) - 64\mathcal{L}(y_2) \\ s\mathcal{L}(y_2) - y_2(0) &= \mathcal{L}(y_1) - 14\mathcal{L}(y_2). \end{aligned}$$

Letting $Y_1 = \mathcal{L}(y_1)$, $Y_2 = \mathcal{L}(y_2)$ and plugging in the initial conditions, we have

$$\begin{aligned} sY_1 - 0 &= 2Y_1 - 64Y_2 \\ sY_2 - 1 &= Y_1 - 14Y_2. \end{aligned}$$

These equations can be written in matrix form as

$$\begin{pmatrix} s-2 & 64 \\ -1 & s+14 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The solution of this system is given by

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \frac{1}{(s-2)(s+14)+64} \begin{pmatrix} s+14 & -64 \\ 1 & s-2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{s^2+12s+36} \begin{pmatrix} -64 \\ s-2 \end{pmatrix}.$$

Now $Y_1(s) = -\frac{64}{(s+6)^2}$, thus

$$y_1(t) = -64\mathcal{L}^{-1}\left(\frac{1}{(s+6)^2}\right) = -64te^{-6t}.$$

Next, $Y_2(s) = \frac{s-2}{(s+6)^2} = \frac{s+6-8}{(s+6)^2} = \frac{1}{s+6} - \frac{8}{(s+6)^2}$, thus

$$y_2(t) = \mathcal{L}^{-1}\left(\frac{1}{s+6}\right) - 8\mathcal{L}^{-1}\left(\frac{1}{(s+6)^2}\right) = e^{-6t} - 8te^{-6t}.$$

5.5 Discontinuous Functions and Periodic Functions

PROBLEMS

In each of Problems 1 through 6, sketch the graph of the given function on the interval $t \geq 0$:

1. $u_1(t) + 3u_3(t) - 7u_4(t)$

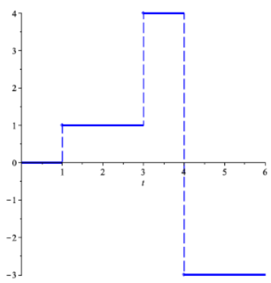
In each of Problems 7 through 12, find the Laplace transform of the given function:

7. $f(t) = \begin{cases} 0, & t < 9 \\ (t-9)^s, & t \geq 9 \end{cases}$

In each of Problems 13 through 18, find the inverse Laplace transform of the given function:

17. $F(s) = \frac{(s-2)e^{-s}}{s^2 - 4s + 3}$

1.



7. Using the Heaviside function, we can write $f(t) = (t-9)^5 u_9(t)$. The Laplace transform has the property that $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t))$. Therefore,

$$\mathcal{L}(u_9(t)(t-9)^5) = e^{-9s}\mathcal{L}(t^5) = \frac{120e^{-9s}}{s^6}.$$

17. Using the fact that $\mathcal{L}^{-1}(e^{-cs}G(s)) = u_c(t)g(t-c)$, we see that

$$\mathcal{L}^{-1}\left(\frac{(s-2)e^{-s}}{s^2-4s+3}\right) = u_1(t)g(t-1),$$

where

$$g(t) = \mathcal{L}^{-1}\left(\frac{s-2}{s^2-4s+3}\right).$$

Using partial fractions, we write

$$\frac{s-2}{s^2-4s+3} = \frac{1}{2}\left[\frac{1}{s-1} + \frac{1}{s-3}\right].$$

Therefore,

$$g(t) = \frac{1}{2}[e^t + e^{3t}],$$

and then

$$\mathcal{L}^{-1}(F(s)) = \frac{1}{2}[e^{t-1} + e^{3(t-1)}]u_1(t).$$

5.6 Differential Equations with Discontinuous Forcing Functions

PROBLEMS

In each of Problems 1 through 13, find the solution of the given initial value problem. Draw the graphs of the solution and of the forcing function; explain how they are related.

1. $y'' + y = f(t); \quad y(0) = 5, \quad y'(0) = 3;$

$$f(t) = \begin{cases} 1, & 0 \le t < \pi/2 \\ 0, & \pi/2 \le t < \infty \end{cases}$$

3. $y'' + 4y = \sin t - u_{2\pi}(t)\sin(t - 2\pi);$
$y(0) = 0, \quad y'(0) = 0$

1. Let $Y(s) = \mathcal{L}(y)$. Applying the Laplace transform to the equation, we have $\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(f(t))$, which is $[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \mathcal{L}(f(t))$. Applying the initial conditions, we get $s^2Y(s) - 5s - 3 + Y(s) = \mathcal{L}(f(t))$. The forcing function $f(t)$ can be written as $f(t) = u_0(t) - u_{\pi/2}(t)$. Therefore,

$$\mathcal{L}(f(t)) = \mathcal{L}(u_0(t)) - \mathcal{L}(u_{\pi/2}(t)) = \frac{1-e^{-\pi s/2}}{s}.$$

Thus the equation for Y becomes

$$[s^2+1]Y(s) = 5s + 3 + \frac{1-e^{-\pi s/2}}{s},$$

and then

$$Y(s) = \frac{5s+3}{s^2+1} + \frac{1-e^{-\pi s/2}}{s(s^2+1)}.$$

Using partial fractions, we write the second term as

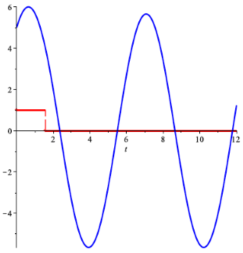
$$(1-e^{-\pi s/2})\left[\frac{1}{s} - \frac{s}{s^2+1}\right].$$

We obtain that

$$Y(s) = \frac{4s+3}{s^2+1} + \frac{1}{s} - \frac{e^{-\pi s/2}}{s} + \frac{se^{-\pi s/2}}{s^2+1}.$$

Then, using the fact that $\mathcal{L}^{-1}(e^{-cs}G(s)) = u_c(t)g(t-c)$, we conclude that

$$y(t) = 4\cos t + 3\sin t + 1 - u_{\pi/2}(t) + u_{\pi/2}(t)\cos(t - \pi/2).$$



3. Let $Y(s) = \mathcal{L}(y)$. Applying the Laplace transform to the equation, we have $\mathcal{L}(y'') + 4\mathcal{L}(y) = \mathcal{L}(\sin(t) - u_{2\pi}(t)\sin(t-2\pi))$, which is

$$[s^2Y(s) - sy(0) - y'(0)] + 4Y(s) = \frac{1 - e^{-2\pi s}}{s^2+1}.$$

Applying the initial conditions, we get

$$s^2Y(s) + 4Y(s) = \frac{1 - e^{-2\pi s}}{s^2+1}.$$

Therefore, the equation for Y becomes

$$[s^2+4]Y(s) = \frac{1 - e^{-2\pi s}}{s^2+1},$$

thus

$$Y(s) = \frac{1 - e^{-2\pi s}}{(s^2+4)(s^2+1)}.$$

Using partial fractions, we can write

$$Y(s) = \frac{1}{3}(1 - e^{-2\pi s})\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right].$$

Therefore, we conclude that

$$y(t) = \frac{1}{3}\left[\sin t - \frac{1}{2}\sin 2t - u_{2\pi}(t)\sin(t-2\pi) + \frac{1}{2}u_{2\pi}(t)\sin(2(t-2\pi))\right].$$