ONLINE RECITATION3 SPRING 2020

Monday, April 13, 2020 11:26 PM

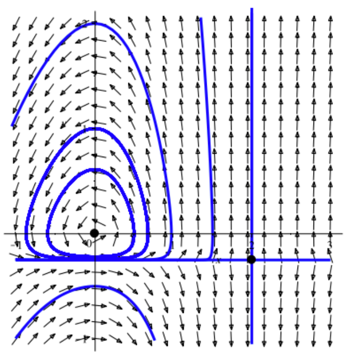
7.1 Autonomous Systems and Stability

For each of the systems in Problems 1 through 18:
(a) Find all the critical points (equilibrium solutions).
(b) Use a computer to draw a direction field and phase portrait for the system.
(c) From the plot(s) in part (b), determine whether each critical point is asymptotically stable, stable, or unstable, and classify it as to type.

1. $dx/dt = -2y + xy, \qquad dy/dt = x + 4xy$

1.(a) $-2y + xy = 0$ implies $y(-2 + x) = 0$ implies $x = 2$ or $y = 0$. Then, $x + 4xy = 0$ implies $x(1 + 4y) = 0$ implies $x = 0$ or $y = -1/4$. Therefore, the critical points are $(2, -1/4)$ and $(0, 0)$.

(b)



(c) The critical point $(0, 0)$ is a center, therefore, stable. The critical point $(2, -1/4)$ is a saddle point, therefore, unstable.

7.2 Almost Linear Systems

In each of Problems 1 through 20:
(a) Determine all critical points of the given system of equations.
(b) Find the corresponding linear system near each critical point.
(c) Find the eigenvalues of each linear system. What conclusions can you then draw about the nonlinear system?

1. $dx/dt = -2x + y, \qquad dy/dt = x^2 - y$

5. $dx/dt = (4 + x)(y - x), \qquad dy/dt = (10 - x)(y + x)$

13. $dx/dt = x - y^2, \qquad dy/dt = y - x^2$

1.(a) The equation $dx/dt = 0$ implies $y = 2x$. The equation $dy/dt = 0$ implies $y = x^2$. Therefore, for these equations to both be satisfied, we need $2x = x^2$ which means $x = 0$ or $x = 2$. Thus the two critical points are $(0, 0)$ and $(2, 4)$.

(b) Here, we have $F(x, y) = -2x + y$ and $G(x, y) = x^2 - y$. Therefore, the Jacobian matrix for this system is

$$\mathbf{J}(x, y) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 2x & -1 \end{pmatrix}.$$

Near the critical point $(0, 0)$, the Jacobian matrix is

$$\mathbf{J}(0, 0) = \begin{pmatrix} F_x(0,0) & F_y(0,0) \\ G_x(0,0) & G_y(0,0) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$$

and the corresponding linear system near $(0, 0)$ is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Near the critical point $(2,4)$, the Jacobian matrix is

$$\mathbf{J}(2,4) = \begin{pmatrix} F_x(2,4) & F_y(2,4) \\ G_x(2,4) & G_y(2,4) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & -1 \end{pmatrix}$$

and the corresponding linear system near $(2,4)$ is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x - 2$ and $v = y - 4$.

(c) The eigenvalues of the linear system near $(0,0)$ are $\lambda = -1, -2$. From this, we can conclude that $(0,0)$ is an asymptotically stable node for the nonlinear system. The eigenvalues of the linear system near $(2,4)$ are $(-3 \pm \sqrt{17})/2$. Since one of these eigenvalues is positive and one is negative, the critical point $(2,4)$ is an unstable saddle point for the nonlinear system.

5.(a) To find the critical points, we need to solve the equations $(4 + x)(y - x) = 0$ and $(10 - x)(y + x) = 0$. Solving this system of equations, we see that the critical points are given by $(0,0)$, $(10,10)$, and $(-4,4)$.

(b) Here, we have $F(x,y) = (4 + x)(y - x)$ and $G(x,y) = (10 - x)(y + x)$. Therefore, the Jacobian matrix for this system is

$$\mathbf{J}(x,y) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} -4 - 2x + y & 4 + x \\ 10 - y - 2x & 10 - x \end{pmatrix}.$$

Near the critical point $(0,0)$, the Jacobian matrix is

$$\mathbf{J}(0,0) = \begin{pmatrix} F_x(0,0) & F_y(0,0) \\ G_x(0,0) & G_y(0,0) \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 10 & 10 \end{pmatrix}$$

and the corresponding linear system near $(0,0)$ is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 10 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Near the critical point $(-4,4)$, the Jacobian matrix is

$$\mathbf{J}(-4,4) = \begin{pmatrix} F_x(-4,4) & F_y(-4,4) \\ G_x(-4,4) & G_y(-4,4) \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 14 & 14 \end{pmatrix}$$

and the corresponding linear system near $(-4,4)$ is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 14 & 14 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x + 4$ and $v = y - 4$. Near the critical point $(10,10)$, the Jacobian matrix is

$$\mathbf{J}(10,10) = \begin{pmatrix} F_x(10,10) & F_y(10,10) \\ G_x(10,10) & G_y(10,10) \end{pmatrix} = \begin{pmatrix} -14 & 14 \\ -20 & 0 \end{pmatrix}$$

and the corresponding linear system near $(10,10)$ is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -14 & 14 \\ -20 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x - 10$ and $v = y - 10$.

(c) The eigenvalues of the linear system near $(0,0)$ are $\lambda = 3 \pm \sqrt{89}$. From this, we can conclude that $(0,0)$ is an unstable saddle point for the nonlinear system. The eigenvalues of the linear system near $(-4,4)$ are $\lambda = 8, 14$. From this, we can conclude that $(-4,4)$ is an unstable node for the nonlinear system. The eigenvalues of the linear system near $(10,10)$ are $\lambda = -7 \pm i\sqrt{231}$. From this, we can conclude that $(10,10)$ is an asymptotically stable spiral point the nonlinear system.

7.3 Competing Species

PROBLEMS

Each of Problems 1 through 6 can be interpreted as describing the interaction of two species with populations x and y. In each of these problems, carry out the following steps.

(b) Find the critical points.
(c) For each critical point, find the corresponding linear system. Find the eigenvalues and eigenvectors of the linear system, classify each critical point as to type, and determine whether it is asymptotically stable, stable, or unstable.
(d) Sketch the trajectories in the neighborhood of each critical point.

1. $dx/dt = x(1.5 - x - 0.5y), \quad dy/dt = y(2 - y - 0.75x)$

2. $dx/dt = x(1.5 - x - 0.5y), \quad dy/dt = y(2 - 0.5y - 1.5x)$

Problem 1.

(b) The critical points are solutions of the system

$$\begin{aligned} x(1.5 - x - 0.5y) &= 0 \\ y(2 - y - 0.75x) &= 0. \end{aligned}$$

The four critical points are $(0,0)$, $(0,2)$, $(1.5,0)$, and $(0.8,1.4)$.

(c) The Jacobian matrix is

$$\mathbf{J}(x,y) = \begin{pmatrix} 3/2 - 2x - y/2 & -x/2 \\ -3y/4 & 2 - 3x/4 - 2y \end{pmatrix}.$$

At $(0,0)$,

$$\mathbf{J}(0,0) = \begin{pmatrix} 3/2 & 0 \\ 0 & 2 \end{pmatrix}.$$

The associated eigenvalues and eigenvectors are $\lambda_1 = 3/2$, $\mathbf{v}_1 = (1,0)^T$ and $\lambda_2 = 2$, $\mathbf{v}_2 = (0,1)^T$. The eigenvalues are positive. Therefore, the origin is an unstable node. At $(0,2)$,

$$\mathbf{J}(0,2) = \begin{pmatrix} 1/2 & 0 \\ -3/2 & -2 \end{pmatrix}.$$

The associated eigenvalues and eigenvectors are $\lambda_1 = 1/2$, $\mathbf{v}_1 = (1,-0.6)^T$ and $\lambda_2 = -2$, $\mathbf{v}_2 = (0,1)^T$. The eigenvalues have opposite sign. Therefore, $(0,2)$ is a saddle, which is unstable. At $(1.5,0)$,

$$\mathbf{J}(1.5,0) = \begin{pmatrix} -1.5 & -0.75 \\ 0 & 0.875 \end{pmatrix}.$$

The associated eigenvalues and eigenvectors are $\lambda_1 = -1.5$, $\mathbf{v}_1 = (1,0)^T$ and $\lambda_2 = 0.875$, $\mathbf{v}_2 = (-0.31579, 1)^T$. The eigenvalues are opposite sign. Therefore, $(1.5,0)$ is a saddle, which is unstable. At $(0.8, 1.4)$,

$$\mathbf{J}(0.8,1.4) = \begin{pmatrix} -0.8 & -0.4 \\ -1.05 & -1.4 \end{pmatrix}.$$

The associated eigenvalues and eigenvectors are $\lambda_1 = (-11+\sqrt{51})/10$, $\mathbf{v}_1 = (1,(3-\sqrt{51})/4)^T$ and $\lambda_2 = (-11-\sqrt{51})/10$, $\mathbf{v}_2 = (1,(3+\sqrt{51})/4)^T$. The eigenvalues are negative. Therefore, $(0.8,1.4)$ is a stable node, which is asymptotically stable.

(d,e)