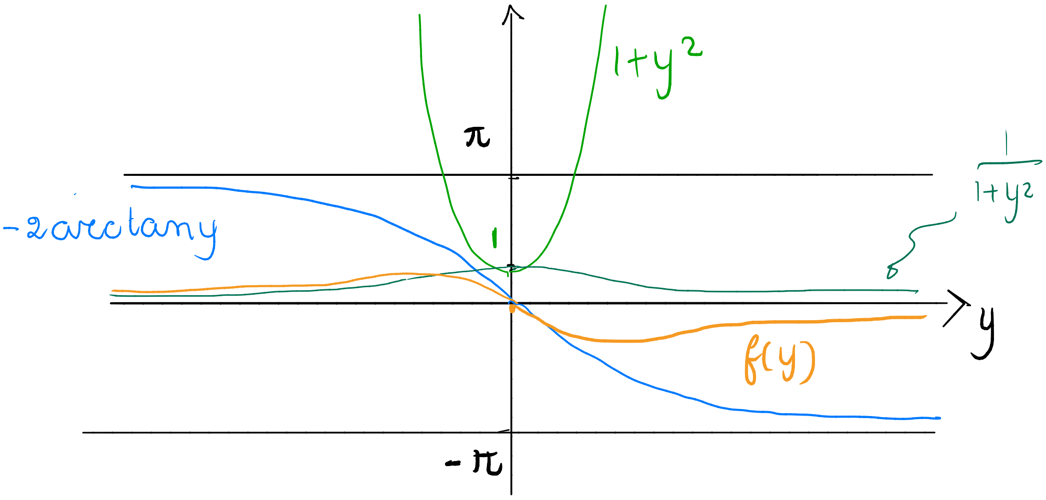
Ex 4

$$\boxed{\frac{dy}{dt} = -2\,\frac{\arctan y}{1+y^2} \qquad -\infty < y_0 < +\infty}$$

$$f(y) = -2\,\frac{\arctan y}{1+y^2}$$

A SKETCH BY HAND (& ITS CONSTRUCTION)

- f is odd (so, its graph is symmetric wrt 0)
- $\lim\limits_{t \to +\infty} f(y) = 0^-$;
- $f(y) = 0 \Leftrightarrow y = 0$; $f < 0$ on $(0, +\infty)$



- Unique critical point $y = 0$.
- $f(y)$ is positive when $y<0$ and negative when $y>0$. Hence $y=0$ is asymptotically stable

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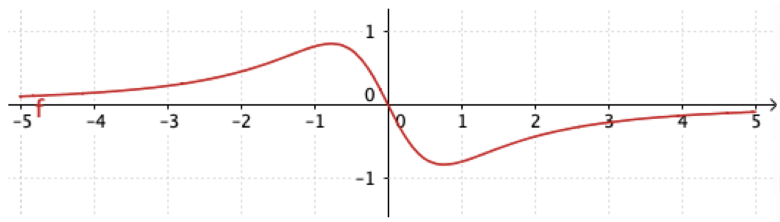
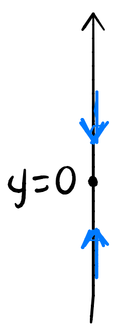
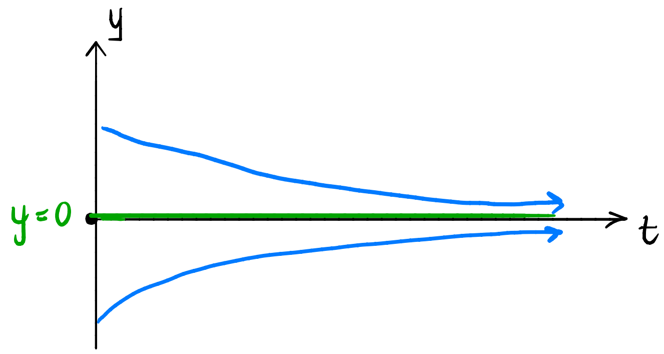




phase line



sketch of some solutions in the ty-plane

Ex 6 $\dfrac{dy}{dt} = ay + by^2$, $a>0, b>0$

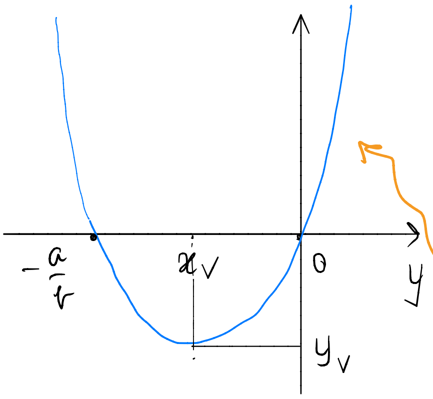
$y_0 > 0$

$f(y) = ay + by^2 = y(a+by)$

The graph of f is a parabola concave upward ($b>0$)

The intersection with the y axis (= solutions of $f(y)=0$) are $y=0$ and $y = -\frac{a}{b}$ (<0)

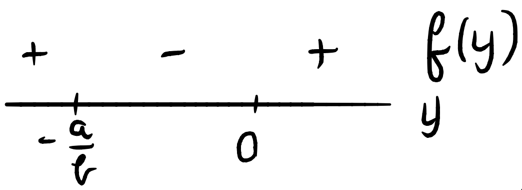
The vertex of the parabola is $V = (x_V, y_V)$ where $\begin{cases} x_V = -\frac{1}{2}\frac{a}{b} \\ y_V = f(x_V) = -\frac{1}{4}\frac{a^2}{b} \end{cases}$



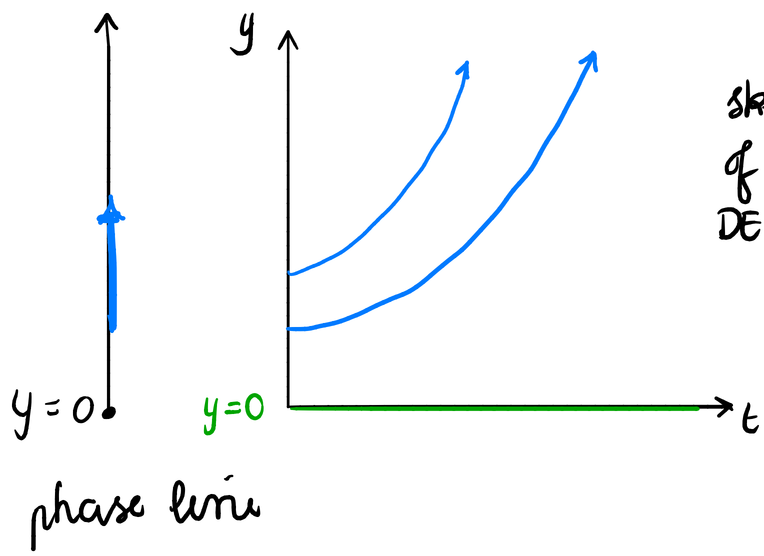
(only the portion above $y>0$ will matter for solutions with initial condition $y_0>0$)

$f(y)=0$ has solutions $y=0$ and $y=-\frac{a}{b}$ (<0). They are equilibrium points of the DE but only $y=0$ plays a role for $y_0>0$

The sign of f is



Hence $y=0$ is asymptotically unstable (and $y=-\frac{a}{b}$ would be asymptotically stable)

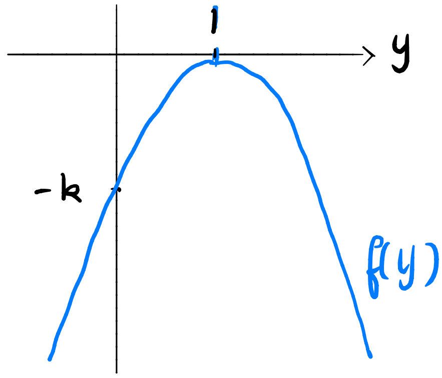


sketch of two solutions of the DE (with $y_0>0$)

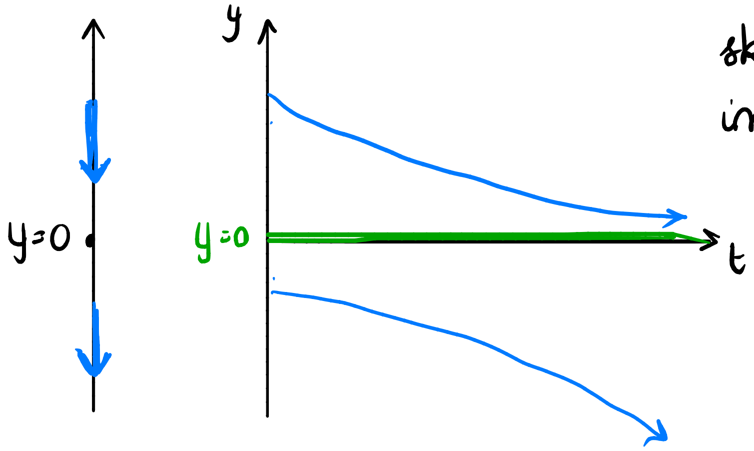
EX 8 $\boxed{\dfrac{dy}{dt} = -k(y-1)^2 \ , \ k>0 \qquad -\infty < y_0 < +\infty}$

$f(y) = -k(y-1)^2$

The graph of f is a parabola concave down $(-k<0)$ and vertex in $V=(1,0)$. It intersects the ordinate axis in $(0,-k)$



- $f(y)=0$ has unique solution $y=1$. So $y=1$ is the unique equilibrium point of the DE
- $f(y)<0$ for all $y \neq 1$. Hence the equilibrium point $y=1$ is asymptotically semistable



phase line

sketch of solutions in the ty-plane

EX 10 $\boxed{\dfrac{dy}{dt} = y(1-y^2) \qquad -\infty < y_0 < \infty}$

$f(y) = y(1-y^2) = y(1-y)(1+y)$

The graph of $f(y)$ is a cubic

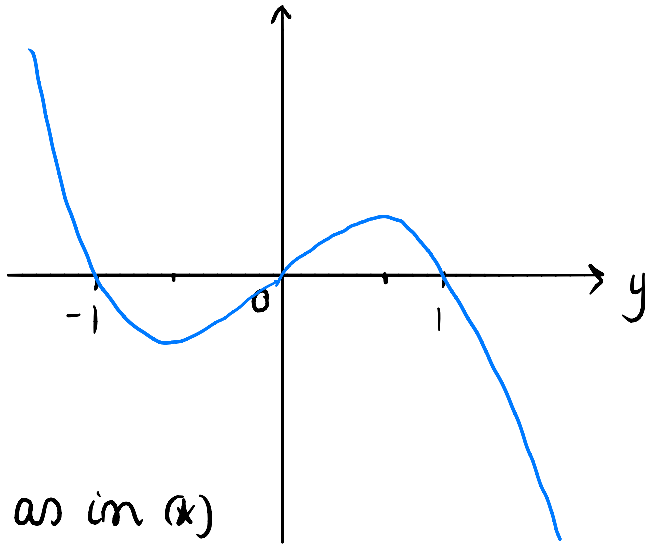
f is odd, so the graph is symm. wrt 0

$f(y)=0 \iff y = 0, 1, -1$

$$\left.\begin{array}{l} f(y)>0 \text{ for } y\in(-\infty,-1)\cup(0,1) \\ \text{and } f(y)<0 \text{ for } y\in(-1,0)\cup(1,+\infty) \end{array}\right\} (*)$$

$f'(y) = 1-y^2-2y^2 = 1-3y^2 \geq 0 \iff y \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

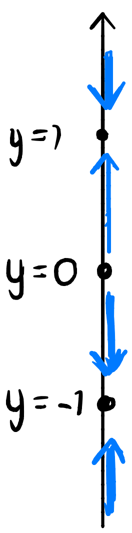
$f\left(\pm\frac{1}{\sqrt{3}}\right) = \pm\frac{1}{\sqrt{3}}\cdot\frac{2}{3}$



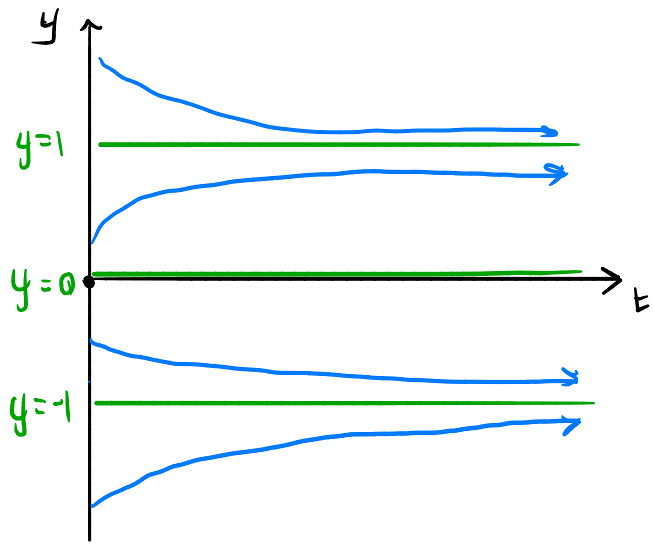
$f(y)=0$ has three solutions: $y = 0, 1, -1$ which are equilibrium points.

Because of the sign of $f(y)$ as in $(*)$

$y=-1$ and $y=1$ are asymptotically stable, whereas $y=0$ is asymptotically unstable



phase line



EX 12

$$\frac{dy}{dt} = y^2(4-y^2)$$
$$-\infty < y_0 < +\infty$$

The equilibrium points (i.e. solutions of $f(y)=0$) are $y=0,2,-2$

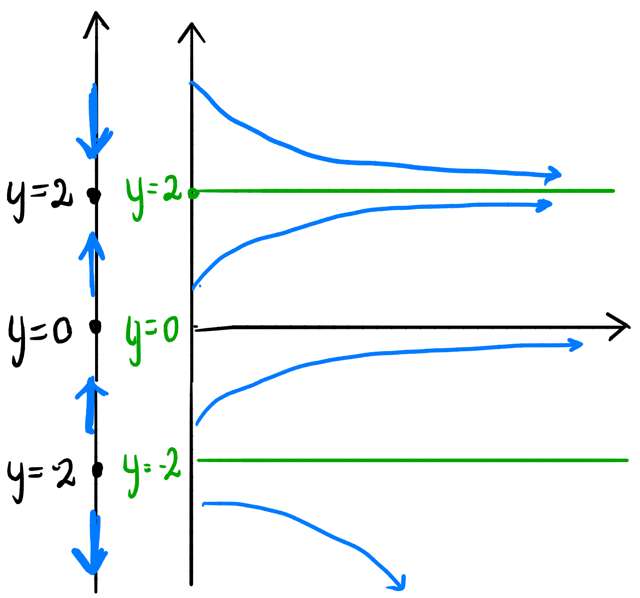
$$\begin{cases} f(y)>0 & \text{if } y\in(-2,0)\cup(0,2) \\ f(y)<0 & \text{if } y<-2 \text{ or } y>2 \end{cases}$$

Therefore:

$y=-2$ is asymptotically unstable

$y=0$ is asymptotically semistable

$y=2$ is asymptotically stable



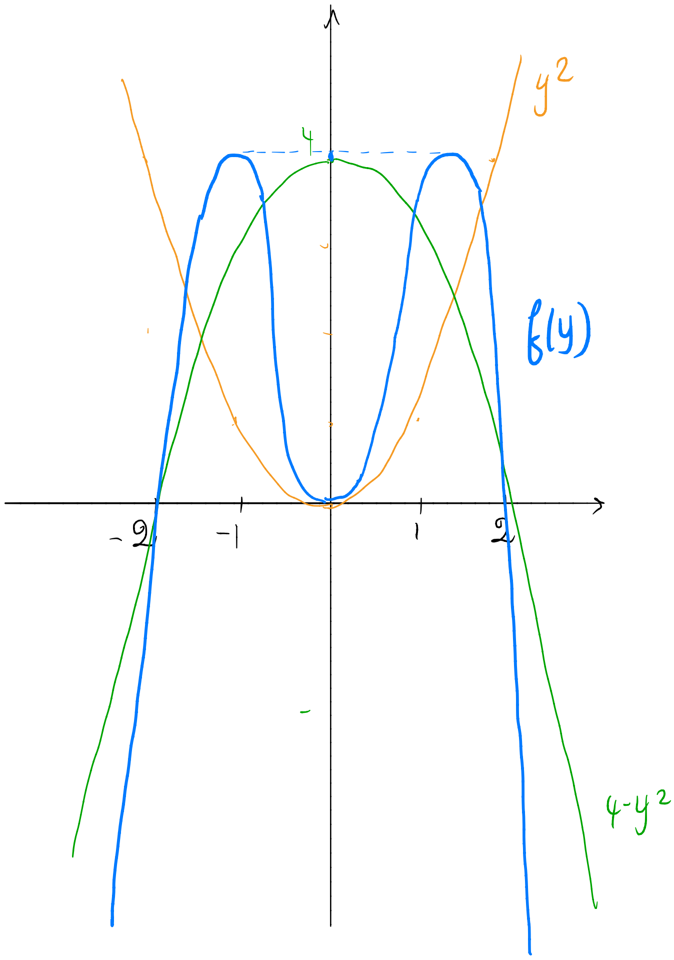
$f(y) = y^2(4-y^2) = y^2(2-y)(2+y)$

The graph of f is a curve symmetric wrt the ordinate axis because f is an even function.

$f(y)=0 \iff y=-2,0,2$

$$\lim_{y\to\pm\infty} f(y) = +\infty$$

FREE HAND SKETCH



(the precise value of the extrema can be found by solving $f'(y)=0$:

$f'(y) = 4y(2-y^2) = 0 \iff y=0,\ y=\pm\sqrt{2}$,

and $f(\pm\sqrt{2}) = 2(4-2) = 4$