Chaotic Scattering on Hyperbolic Manifolds

Peter A. Perry

University of Kentucky

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1. Introduction

Suppose that X is the (non-compact) quotient of real hyperbolic space $\mathbb{H}^{n+1}$ by a geometrically finite, discrete group of hyperbolic isometries

- X has a chaotic (Anosov) geodesic flow induced from the geodesic flow on $\mathbb{H}^{n+1}$
- X has a Laplacian induced from the Laplacian on $\mathbb{H}^{n+1}$ which describes quantum scattering on X
- Attached to X is a Selberg zeta function that $Z_\Gamma(s)$ which links the length spectrum of geodesics with spectral data of the Laplacian

These features make such manifolds X an excellent "laboratory" to study chaotic scattering

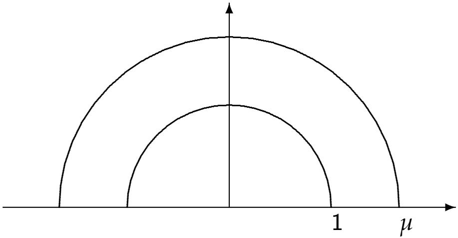
Example I: The Hyperbolic Cylinder (1 of 4)

Consider the discrete group of dilations

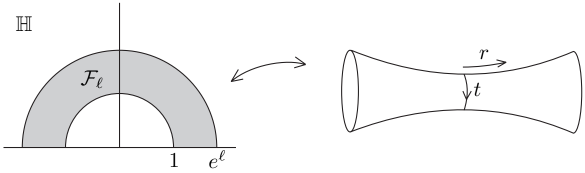
$$z \mapsto \mu^n z$$

acting on the upper half plane with Poincaré metric

$$ds^2 = y^{-2}\left(dx^2 + dy^2\right)$$



The Hyperbolic Cylinder (2 of 4)

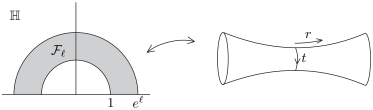


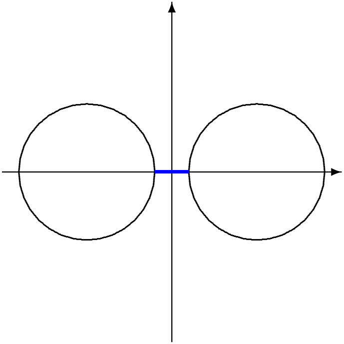
The quotient $X = \mathbb{H}/\Gamma$ is a hyperbolic funnel $\mathbb{R} \times S^1$ with metric

$$ds^2 = dr^2 + \ell^2 \cosh^2 r \, dt^2$$

and a single closed geodesic of length $\ell = \log \mu$

The Hyperbolic Cylinder (3 of 4)





Hyperbolic Cylinder

Two Convex Obstacles

$$(X, g_0) \simeq \left(\mathbb{R} \times S^1, dr^2 + \ell^2 \cosh^2 r \, dt^2 \right)$$

The Hyperbolic Cylinder (4 of 4)

- The length spectrum is $\{\ell\}$ and there is a single, unstable, closed geodesic
- The Laplacian is separable and its resolvent may be computed using special functions
- The Selberg zeta function

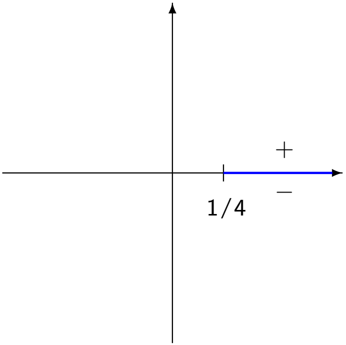
$$Z_\Gamma(s) = \prod_{k=1}^{\infty} \left(1 - e^{-(s+k)\ell(\gamma)}\right)$$

has a lattice of zeros at

$$s_{n,k} = -k + \frac{2\pi i n}{\ell}$$

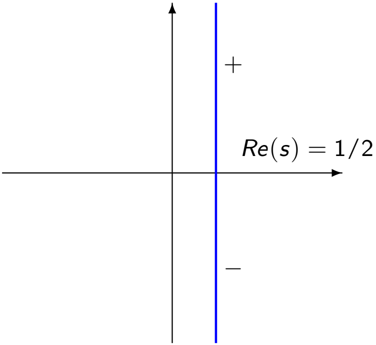
for $k = 0, 1, 2, \ldots$ and $n \in \mathbb{Z}$.

Resonances of the Hyperbolic Cylinder (1 of 2)



$(\Delta_X - \lambda)^{-1}$

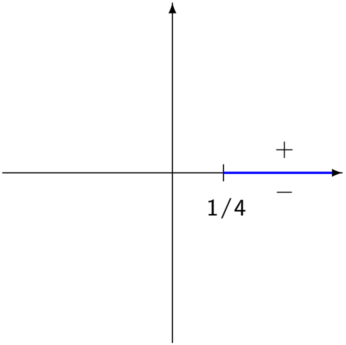
Complex λ-plane



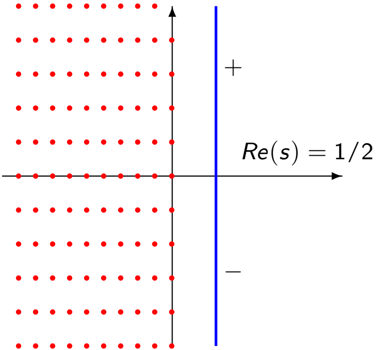
$(\Delta_X - s(1-s))^{-1}$

Complex s-plane

Resonances of the Hyperbolic Cylinder (2 of 2)



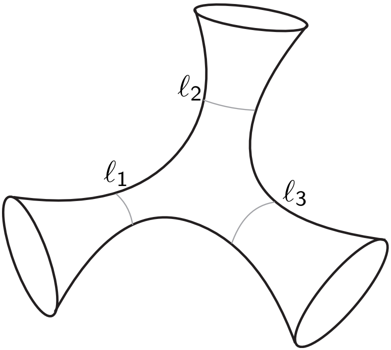
$(\Delta_X - \lambda)^{-1}$



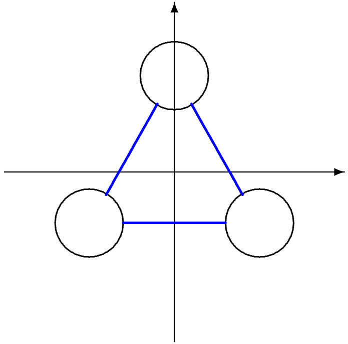
$(\Delta_X - s(1-s))^{-1}$

Poles at $s_{n,k} = -k + \dfrac{2\pi i n}{\ell}$

Example II: A Pair of Trousers with Hyperbolic Ends (1 of 2)

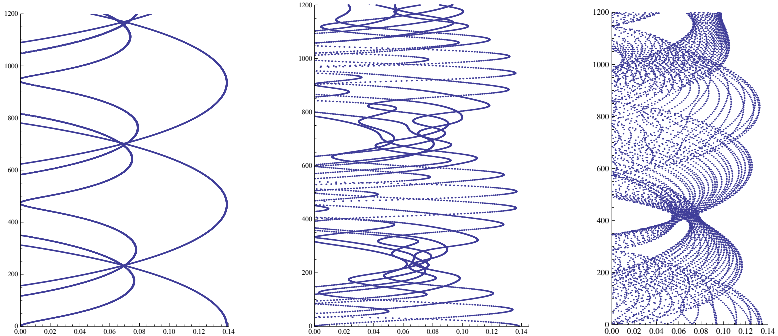


$X(\ell_1, \ell_2, \ell_3)$



Three convex obstacles

Trousers with Hyperbolic Ends (2 of 2)



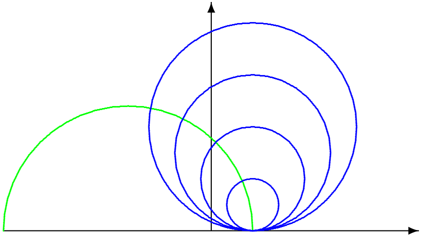
$X(10,10,10)$ $X(10,10,10.1)$ $X(10,10,11)$

Resonances of 3-funnel surfaces $X(\ell_1, \ell_2, \ell_3)$, consisting of funnels attached to a hyperbolic pair of pants with boundary lengths ℓ_1, ℓ_2, ℓ_3.

From David Borthwick, Distribution of resonances for hyperbolic surfaces. *Exp. Math.* **23** (2014), no. 1, 25–45.

Geometry of $\mathbb{H}$

- $\left(\mathbb{R}^2_+, y^{-2}\left(dx^2 + dy^2\right)\right)$, $\partial_\infty \mathbb{H} = \mathbb{R} \cup \{\infty\}$
- $\left(\mathbb{B}, 4|dz|^2/(1-|z|^2)^2\right)$, $\partial_\infty \mathbb{H} = S^1$
- Isometries $\mathrm{PSL}(2,\mathbb{R})$ or $\mathrm{PSU}(1,1)$
- Geodesics are semicircles that intersect the boundary normally
- Wavefronts are horocycles



Geodesic Flow

Consider the disc model of $\mathbb{H}$ with $\partial_\infty \mathbb{B} = S^1$. Let

$$(S^1 \times S^1)_- = \{(z_-, z_+) \in S^1 \times S^1 : z_- \neq z_+\}$$

The unit tangent bundle $S\mathbb{H}$ is identified with $(S^1 \times S^1)_- \times \mathbb{R}$ as follows. For $(z_-, z_+, s) \in (S^1 \times S^1)_- \times \mathbb{R}$:

1. Let $[z_-, z_+]$ be the oriented geodesic from z_- to z_+
2. Let s be the signed (hyperbolic) arc length along this geodesic, with $s = 0$ corresponding to the Euclidean midpoint
3. Identify (z_-, z_+, s) with the tangent vector along $[z_-, z_+]$ at this point

In these coordinates, geodesic flow is

$$(z_-, z_+, s) \mapsto (z_-, z_+, s + t)$$

Discrete Groups of Isometries

In the upper half-plane model, the isometries

$$z \mapsto \frac{az+b}{cz+d}$$

are isomorphic to the group $PSL(2, \mathbb{R})$ and are of three types:

Name	Characterization	Example
Elliptic	Rotation	$z \mapsto -1/z$
Parabolic	Translation	$z \mapsto z+1$
Hyperbolic	Dilation	$z \mapsto \mu z$

A *discrete group* Γ of isometries of $\mathbb{H}$ is a group which is topologically discrete as a subset of $PSL(2, \mathbb{R})$.

Discrete Groups, Fundamental Domain

If Γ is a discrete group, the orbit space $X = \mathbb{H}/\Gamma$ is

- an orbifold if Γ has elliptic elements
- a smooth manifold if Γ has no elliptic elements.

A *fundamental domain* for Γ is a closed subset $\mathcal{F}$ of $\mathbb{H}$ so that

(a) $\cup_{\gamma \in \Gamma} \gamma(\mathcal{F}) = \mathbb{H}$

(b) The interiors of $\mathcal{F}$ and $\gamma(\mathcal{F})$ have empty intersection for all $\gamma \neq e$

A discrete group Γ is *geometrically finite* if Γ admits a finite-sided fundamental domain $\mathcal{F}$

The Limit Set (1 of 2)

If Γ is a discrete group, the *limit set* of Γ is the set of accumulation points of Γ-orbits on $\partial_\infty \mathbb{H}$

The complement of the limit set in $\partial_\infty \mathbb{H}$ is the *ordinary set* $\Omega(\Gamma)$.

Theorem (Poincaré, Klein-Fricke) *For a discrete subgroup* Γ *of* $\mathrm{Isom}(\mathbb{H})$*, the limit set* $\Lambda(\Gamma)$ *is either*

(a) *0, 1, or 2 points, if* Γ *is elementary,*
(b) *A nowhere dense, perfect subset of* $\partial_\infty \mathbb{H}$*, or*
(c) *All of* $\partial_\infty \mathbb{H}$

The Limit Set (2 of 2)

Theorem (Poincaré, Klein-Fricke) *For a discrete subgroup* Γ *of* $\mathrm{Isom}(\mathbb{H})$, *the limit set* $\Lambda(\Gamma)$ *is either*

(a) *0, 1, or 2 points, if* Γ *is elementary,*

(b) *A nowhere dense, perfect subset of* $\partial_\infty \mathbb{H}$, *or*

(c) *All of* $\partial_\infty \mathbb{H}$

The group generated by $z \mapsto z + 1$ or $z \mapsto \mu z$ are elementary.

If $\mathbb{H}/\Gamma$ is compact or has finite volume, $\Lambda(\Gamma) = \partial_\infty \mathbb{H}$.

What lies in between?

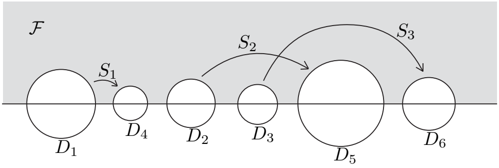
Schottky Groups and their Quotients (1 of 4)

A *Schottky group* is a discrete group Γ with a certain geometrically described set of generators.

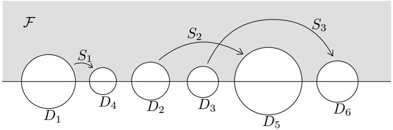
Suppose that $\{D_1, \cdots, D_{2r}\}$ are a collection of open Euclidean discs in $\mathbb{C}$ with disjoint closures and centers on the real axis.

Let $S_j \in \mathrm{Isom}(\mathbb{H})$ map ∂D_j to ∂D_{j+r} and $\mathrm{ext}(D_j)$ to $\mathrm{int}(D_{j+r})$. Order the indices so that

$$S_{j+2r} = S_j, \quad S_{j+r} = S_j^{-1}$$



Schottky Groups and their Quotients (2 of 4)



Let $S_j \in \mathrm{Isom}(\mathbb{H})$ map ∂D_j to ∂D_{j+r} and $\mathrm{ext}(D_j)$ to $\mathrm{int}(D_{j+r})$. Order the indices so that

$$S_{j+2r} = S_j, \quad S_{j+r} = S_j^{-1}$$

A discrete group Γ is a *Schottky group* if there is a set of discs $\{D_j\}_{j=1}^{2r}$ so that Γ is generated by the transformations $\{S_j\}_{j=1}^{2r}$.

A discrete group Γ is *convex co-compact* if the fundamental domain for Γ does not touch the limit set $\Lambda(\Gamma)$. Button proved that all convex co-compact discrete groups in $\mathbb{H}$ are Schottky.

Schottky Groups and their Quotients (3 of 4)

Suppose Γ is a Schottky group associated to open discs $\{D_1, \cdot, D_{2r}\}$ and generated by $\{S_1, \cdots S_{2r}\}$.

- The region $\mathcal{F} = \mathbb{H} - \cup_{j=1}^{2r} D_j$ is a fundamental domain for Γ
- $X = \mathbb{H}/\Gamma$ is a hyperbolic manifold of infinite volume and genus $1 - r$

The geodesic flow on $X = \mathbb{H}/\Gamma$ can be coded by the *Bowen-Series Map*. Let $I_j = D_j \cap \mathbb{R}$ and define

$$B : \cup_{j=1}^{2r} I_j \to \cup_{j=1}^{2r} I_j$$

by

$$Bq = S_j q, \quad q \in I_j$$

Associated to such maps is a dynamical zeta function which will play an important role later.

Schottky Groups and their Quotients (4 of 4)

The geodesic flow on $X = \mathbb{H}/\Gamma$ can be coded by the *Bowen-Series Map*. Let $I_j = D_j \cap \mathbb{R}$ and define

$$B : \cup_{j=1}^{2r} I_j \to \cup_{j=1}^{2} rI_j$$

by

$$Bq = S_j q, \quad q \in I_j$$

There is a one-to-one correspondence between primitive periodic orbits $\{q, Bq, \cdots B^n q\}$ of B and primitive closed geodesics of X having length $\ell = \log |(B^n)'(q)|$.

Hausdorff Dimension of the Limit Set

A natural object that measures the 'complexity' of the limit set is the *exponent of convergence* δ for the Poincaré series

$$P(z, z'; s) = \sum_{\gamma \in \Gamma} e^{-s d(z, \gamma(z'))}$$

where $d(\,\cdot\,,\,\cdot\,)$ is hyperbolic distance.

Theorem (Patterson-Sullivan) *The exponent of convergence δ is the Hausdorff dimension of $\Lambda(\Gamma)$.*

Important Fact: $s = \delta$ gives the lowest eigenvalue (if $\delta > 1/2$) or the first resonance (if $\delta < 1/2$)

Trapped Orbits on $X = \mathbb{H}/\Gamma$

Important Fact: The *trapped set* for geodesic flow has Hausdorff dimension $1 + 2\delta$ in the unit tangent bundle.

Recall that

$$S\mathbb{H} \simeq (S^1 \times S^1)_- \times \mathbb{R}$$

Trapped orbits in SX are identified with closed geodesics whose endpoints lie in the limit set

Note that for $\delta = 1$ (compact or finite-volume) the trapped set has full dimension.

Conformal Compactification, 0-integral, 0-trace

If $X = \mathbb{H}/\Gamma$ and Γ is convex co-compact, then X compactifies to a manifold with boundary, $\overline{X}$. In our case, this coincides with the Klein compactification of $\mathbb{H}/\Gamma$ to $(\mathbb{H}/\Gamma) \cup (\Omega(\Gamma)/\Gamma))$

If ρ is a defining function for $\partial\overline{X}$, the hyperbolic metric g on X takes the form $g = \rho^{-2}h$ where h is a smooth metric on $\overline{X}$. Such a manifold is called a *conformally compact manifold*.

On a conformally compact manifold, the 0-integral of a smooth function f is

$$^0\int_X f = \mathrm{FP}_{\varepsilon\downarrow 0} \int_{\rho>\varepsilon} f\, dg$$

and the 0-trace of an operator with smooth kernel is the 0-integral of the kernel on the diagonal. The 0-volume of X is the 0-integral of 1.

Scattering Theory

Let (X, g) be a Riemannian manifold, Δ_X its positive Laplacian, and consider the Cauchy problem

$$u_{tt} + (\Delta_X - 1/4)\, u = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

Since Δ_X is self-adjoint, the formal solution is

$$u(t) = \cos\left(t\sqrt{\Delta_X - 1/4}\right) f + \frac{\sin\left(t\sqrt{\Delta_X - 1/4}\right)}{\sqrt{\Delta_X - 1/4}} g$$

We construct functions of a self-adjoint operator A via *Stone's formula*

$$f(A) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{1}{A - \lambda - i\varepsilon} - \frac{1}{A - \lambda + i\varepsilon} \right) f(\lambda)\, d\lambda$$

Stone's formula shifts attention to the *resolvent*

$$R_X(\lambda) = (\Delta_X - \lambda)^{-1}$$

The Resolvent

$$R_X(\lambda) = (\Delta_X - \lambda)^{-1}$$

Suppose $X = \mathbb{H}/\Gamma$ where Γ has no elliptic elements and is geometrically finite.

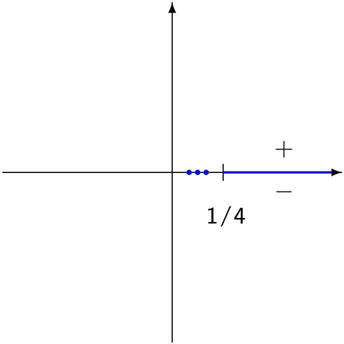
X	Spectrum	Resolvent $(\Delta_X - \lambda)^{-1}$
Compact	Discrete Spectrum	Meromorphic in $\mathbb{C}$
Not compact, Finite Volume	Discrete in $[0, 1/4)$ Continuous in $[1/4, \infty)$	Meromorphic in $\mathbb{C} \setminus [1/4, \infty)$ Resonances in a strip
Infinite volume	Discrete in $[0, 1/4)$ Continuous in $[1/4, \infty)$	Meromorphic in $\mathbb{C} \setminus [1/4, \infty)$ Resonances in a half-plane

Discrete spectrum yields bound states and “confined” motion

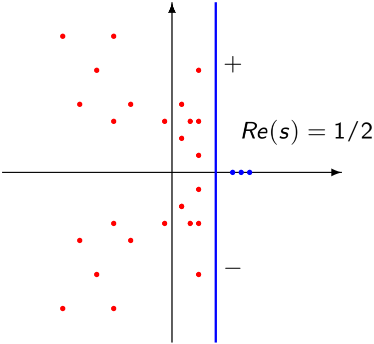
Continuous spectrum corresponds to scattering

Scattering resonances give localized states that “leak out”

Analytic Continuation of the Resolvent (1 of 2)



$(\Delta_X - \lambda)^{-1}$



$(\Delta_X - s(1-s))^{-1}$

Analytic Continuation of the Resolvent (2 of 2)

Let $R_X(s) = (\Delta_X - s(1-s))^{-1}$

The resolvent is $R_X(s) : L^2(X) \to L^2(X)$ is analytic on $\Re(s) > 1/2$ except for finitely many poles $\zeta \in [0, 1/4)$ where $\zeta(n - \zeta)$ is an eigenvalue

The resolvent $R_X(s) : \mathcal{C}_0^\infty(X) \to \mathcal{C}^\infty(X)$ (i.e., the integral kernel of the resolvent) has a meromorphic continuation to the complex s-plane

Structure of the Resolvent Kernel (1 of 3)

To describe the resolvent kernel, consider $\mathbb{H} \times \mathbb{H}$ (upper half-space model) with coordinates (x, y, x', y') let

$$\tau = \sqrt{(x - x')^2 + y^2 + (y')^2},$$

and let

$$(\omega, \eta, \eta') = \frac{(x - x', y, y')}{\tau}$$

The resolvent kernel on $\mathbb{H}$ is a function of the point-pair invariant

$$\sigma(x, y, x', y') = \frac{1}{2} + \frac{(x - x')^2 + y^2 + (y')^2}{4yy'} = \frac{1 + 2\eta\eta'}{4\eta\eta'}$$

Structure of the Resolvent Kernel (2 of 3)

$$\tau = \sqrt{(x-x')^2 + y^2 + (y')^2},$$

$$(\omega, \eta, \eta') = \frac{(x-x', y, y')}{\sqrt{(x-x')^2 + y^2 + (y')^2}}$$

The map $(\tau, x, \omega, \eta, \eta') \mapsto (x, y, x', y')$ is smooth but note the pre-image of $(x, 0, x, 0)$ is a quarter-sphere $S^2_{++} = (0, x, \omega, \eta, \eta')$

The coordinates $(\tau, x, \omega, \eta, \eta')$ describe a blow-up of $\mathbb{H} \times \mathbb{H}$ along the the diagonal of the 'corner' $y = y' = 0$. This blowup is needed to describe the structure of the resolvent kernel.

Structure of the Resolvent Kernel (3 of 3)

Let $\overline{X} \times_0 \overline{X}$ be the corresponding blow-up of $\overline{X} \times \overline{X}$ to a manifold with corners.

Let ρ and ρ' be defining functions for $\partial\overline{X}$ in the first and second factors.

Theorem (Mazzeo-Melrose) *If X is convex co-compact then*

$$R_X(\cdot\,,\,\cdot\,; s) \in \mathcal{I}_0^{-2}(\overline{X} \times_0 \overline{X}) + (\eta\eta')^s \mathcal{C}^\infty(\overline{X} \times_0 \overline{X}) + (\rho\rho')^s \mathcal{C}^\infty(\overline{X} \times \overline{X})$$

with meromorphy in $s \in \mathbb{C}$.

Analytic Structure of the Resolvent (1 of 2)

$R_X(s)$ is *finitely meromorphic*: that is, near each resonance,

$$R_X(s) = \sum_{j=1}^{p} \frac{A_j(\zeta)}{(s(1-s) - \zeta(1-\zeta))^j} + H(s)$$

where $H(s)$ is a holomorphic operator-valued function near $s = \zeta$ and $A_j(\zeta)$ are finite-rank operators with smooth integral kernels.

Analytic Structure of the Resolvent (2 of 2)

$R_X(s)$ is *finitely meromorphic*: that is, near each resonance,

$$R_X(s) = \sum_{j=1}^{p} \frac{A_j(\zeta)}{(s(1-s) - \zeta(1-\zeta))^j} + H(s)$$

where $H(s)$ is a holomorphic operator-valued function near $s = \zeta$ and $A_j(\zeta)$ are finite-rank operators with smooth integral kernels.

The *multiplicity* of a resonance $\zeta \in \mathbb{C}$, $\Re(\zeta) < 1/2$

$$m(\zeta) := \operatorname{rank}(A_1(\zeta))$$

where γ_ζ is a positively oriented circle containing ζ and no other resonance.

We denote by $\mathcal{R}_X$ the resonance set of Δ_X

Resonance Wave Expansions (1 of 2)

Using

$$R_X(s) = (\Delta_X - s(1-s))^{-1}$$

we compute the solution operator for the wave equation

$$\begin{aligned} \cos\left(t\sqrt{\Delta_X - 1/4}\right) &= \frac{1}{2\pi i}\int_{\Re(s)=\frac{1}{2}} \Im\left[(\Delta_X - s(1-s))^{-1}\right](2s-1)\,ds \\ &+ \sum_{j=1}^{N} \cosh\left(t\sqrt{1/4-\lambda_j}\right)P_j \end{aligned}$$

The contribution from resonances comes from "shifting the contour" to a line $\Re(s) = -N$

Resonance Wave Expansions (2 of 2)

Theorem (Christiansen-Zworski) *Suppose* $X = \mathbb{H}/PSL(2,\mathbb{Z})$, *and* $\chi, f \in \mathcal{C}_0^\infty(X)$. *Then for any* N,

$$\chi \frac{\sin t\sqrt{\Delta_X - 1/4}}{\sqrt{\Delta_X - 1/4}} f = \frac{1}{2i} \sum_{\lambda \in \sigma_p(\Delta_X)} \left(\frac{e^{i\sqrt{\lambda_j - 1/4}t} - e^{-i\sqrt{\lambda_j - 1/4}t}}{\sqrt{\lambda_j - 1/4}} \chi(z) C_j(f)(z) \right)$$

$$+ \sum_{s_j \in \mathcal{R}_X} e^{(s_j - 1/2)t \,\mathrm{sign}(1/2 - \Re(s_j))} \sum_{k \leq m(s_j) - 1} v_{jk}(f)(z) t^k$$

$$+ \mathcal{O}\left(e^{-tN}\right)$$

Poisson Formula (Guillopé-Zworski)

Suppose X has c cusps with boundary length h_i, let $\mathcal{P}$ be the collection of prime geodesics $\mathcal{C}$, and let $P_{\mathcal{C}}$ be the Poincaré map for $\mathcal{C}$. As distributions on $\mathbb{R}$,

$$
\begin{aligned}
\text{0-tr} \cos t\sqrt{\Delta_X - \frac{1}{4}} \quad = \quad & -\frac{0 - \mathrm{Vol}(X)}{8\pi} \frac{\cosh(t/2)}{\sinh^2(t/2)} \\
& + \frac{1}{2} \sum_{\mathcal{C} \in \mathcal{P}} \sum_{k=1}^{\infty} \frac{\ell(\mathcal{C})}{|1 - P_{\mathcal{C}}^k|^{1/2}} \delta\left(|t| - k\ell(\mathcal{C})\right) \\
& + \frac{c}{4} \coth(|t|/4) \\
& + \left[c(\gamma - \log 2) - \sum_{i=1}^{c} \log h_i \right] \delta(t)
\end{aligned}
$$

Selberg's Zeta Function

Suppose that Γ is geometrically finite and has only hyperbolic elements (so $X = \mathbb{H}/\Gamma$ is a smooth manifold without cusps).

Each conjugacy class $\{\gamma\}$ corresponds to a closed geodesic of X.

Call a geodesic *prime* if it is not a power of any other closed geodesic. Denote by $\ell(\gamma)$ the length of γ and by $\mathcal{P}$ the set of prime geodesics

Selberg's Zeta function is given by

$$\begin{aligned} Z_X(s) &= \prod_{\gamma \in \mathcal{P}} \prod_{k=1}^{\infty} \left(1 - e^{-(s+k)\ell(\gamma)}\right) \\ &= \exp\left(\sum_{\gamma \in \mathcal{P}} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \frac{e^{-s\ell(\gamma^m)}}{1 - e^{\ell(\gamma^m)}} \right) \end{aligned}$$

Selberg's Zeta Function

Selberg's zeta function plays a central role in the study or resonances because

- It can be connected to dynamical zeta functions and its analyticity properties elucidated using dynamical methods (Ruelle-Fried, Patterson, Pollicott, Naud, ...)
- It can be connected to the spectral theory of the Laplace operator through the trace formula (Patterson, Patterson-Perry, Guillopé-Zworski, Guillopé-Zworski-Lin,...)

Analytic Continuation (Zworski, Guillopé-Lin-Zworski)

If Γ is a Schottky group associated to open discs $\{D_1, \cdots, D_{2r}\}$ and generated by isometries $\{S_1, \cdots, S_{2r}\}$,

Selberg's zeta function for a Schottky group Γ can be represented as a dynamical zeta function associated to the Bowen-Series map.

Let

$$U = \cup_{j=1}^{2r} D_j$$

and

$$\mathcal{H}(U) = \{u \in L^2(U) : u \text{ is analytic on } U\}$$

Recall $Bq = S_j q$ for $q \in I_j = D_j \cap \mathbb{R}$ and extend B to U by setting $B|_{D_j} = S_j$.

Analytic Continuation (Zworski, Guillopé-Lin-Zworski)

$$U = \cup_{j=1}^{2r} D_j, \quad \mathcal{H}(U) = \left\{ u \in L^2(U) : u \text{ is analytic on } U \right\}$$

The *Ruelle Transfer Operator* is the map $L(s) : \mathcal{H}(U) \to \mathcal{H}(U)$ defined by

$$L(s)u(z) = \sum_{w \in U : Bw = z} B'(w)^{-s} u(w)$$

and the *dynamical zeta function* associated to $L(s)$ is

$$d_X(s) = \det(I - L(s))$$

As $L(s)$ is a trace-class operator-valued analytic function $d_X(s)$ is entire. A computation using the holomorphic Lefschetz fixed point formula shows that

$$Z_X(s) = d_X(s)$$

Divisor of Selberg's Zeta Function (1 of 6)

Let $G_\infty(s) = \Gamma(s)G(s)^2$ where $G(s)$ is Barnes' double gamma function. (poles at $s = -n$, multiplicity $2n+1$, $n = 0, 1, \cdots$)

Using scattering theory we can compute the divisor in terms of scattering resonances and topological data of X

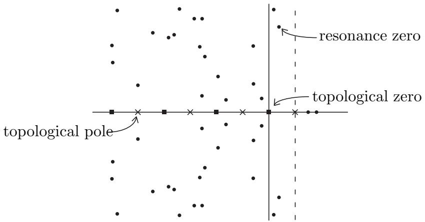
Theorem *If Γ is convex co-compact then*

$$Z_\Gamma(s) = e^{q(s)} P_X(s) G_\infty(s)^{-\chi(X)}$$

where $q(s)$ is a polynomial of degree at most 2, and $P_X(s)$ is an entire function whose zeros (with multiplicity) are determined by the resonance set of Δ_X.

Divisor of Selberg's Zeta Function (2 of 6)

- Topological zeros of multiplicity $(2n+1)(-\chi(X))$ at $s=-n$, $n = 0, 1, 2 \cdots$
- Spectral zeros at $s=\zeta$ where $\zeta > 1/2$ and $\zeta(1-\zeta)$ ia an eigenvalue of the Laplacian, with the multiplicity of the eigenvalue
- Spectral zeros at $s=\zeta$ with multiplicity m_ζ for each resonance



The first zero of $Z_X(s)$ occurs at $s=\delta$

In case X has cusps, $Z_\Gamma(s)$ also has poles

Divisor of Selberg's Zeta Function (3 of 6)

Theorem *If Γ is convex co-compact then*

$$Z_\Gamma(s) = e^{q(s)} P_X(s) G_\infty(s)^{-\chi(X)}$$

where $q(s)$ is a polynomial of degree at most 2, and $P_X(s)$ is an entire function whose zeros (with multiplicity) are determined by the resonance set of Δ_X.

Ideas of the proof:

- $d_X(s) = \det(I - L(s))$ is an entire function of order 2 by estimates on singular values of $L(s)$
- $Z_X(s) = d_X(s)$
- $Z_X(s)$ obeys a functional equation determined by topological and scattering data

Divisor of Selberg's Zeta Function (4 of 6)

Let $\pi : \mathbb{H} \to X$ be the natural projection. Using the identity

$$R_X(\pi(z), \pi(z'); s) = \sum_{\gamma \in \Gamma} R_{\mathbb{H}}(z, \gamma(z'); s)$$

one can show that for $\Re(s) > 1$ and $\mathcal{F}$ a fundamental domain for $\Gamma \in \mathbb{H}$,

$$\frac{Z_X'(s)}{Z_X(s)} = (2s-1) \int_{\mathcal{F}} \Phi(z; s) \, dA(z)$$

where

$$\Phi(z; s) = \left. \left(R_X(\pi(z), \pi(w); s) - R_{\mathbb{H}}(z, w; s) \right) \right|_{z=w}$$

Divisor of Selberg's Zeta Function (5 of 6)

This expression still makes sense on the line $\Re(s) = 1/2$ if we take

$$\Upsilon(s) := \frac{Z_X'(s)}{Z_X(s)} = (2s-1)\,{}^0\!\!\int_{\mathcal{F}} \Phi(z;s)\, dA(z)$$

where

$$\Phi(z;s) = \left(R_X(\pi(z),\pi(w);s) - R_{\mathbb{H}}(z,w;s)\right)\big|_{z=w}$$

since, by the structure of the resolvent kernel,

$$\Phi(z;s) = y^{2s} F(x,y;s)$$

in local coordinates (x,y), where F is a smooth function.

Divisor of Selberg's Zeta Function (6 of 6)

This identity leads to a functional equation on the line $\Re(s) = 1/2$:

$$\begin{aligned}\frac{Z_X'(s)}{Z_X(s)} + \frac{Z_X'(1-s)}{Z_X(1-s)} &= \Upsilon(s) + \Upsilon(1-s) \\ &\quad -(2s-1)\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(s-1/2)\Gamma(1/2-s)}\chi(X)\end{aligned}$$

The first right-hand term gives rise to zeros from the resonances, while the second right-hand term gives rise to topological zeros.

5. Resonances: Theorems and Questions

Let

$$N_X(r) = \#\{\zeta \in \mathcal{R}_X : |\zeta - 1/2| \leq r\}$$

How does this counting function reflect the nature of the trapped set of geodesics?

Theorem (Guillopé Zworski) *Suppose that* $X = \mathbb{H}/\Gamma$ *where* Γ *is geometrically finite, and* X *is non-compact. Then* $N_X(r) \asymp C_X r^2$.

- This result is due to Guillopé-Zworski using techniques of scattering theory including Fredholm determinants for the upper bound and the Poisson summation formula for resonances for the lower bound. Their result is robust under compact perturbations of X.

Theorem (Guillopé-Zworski) *Suppose that* $X = \mathbb{H}/\Gamma$ *where* Γ *is geometrically finite, and* X *is non-compact. Then* $N_X(r) \asymp C_X r^2$.

- The upper bound may be deduced, in the convex co-compact case, from the fact that $Z_X(s)$ is entire of order 2. In higher dimensions, the zeta function can be used to deduce upper and lower bounds (with some important caveats).

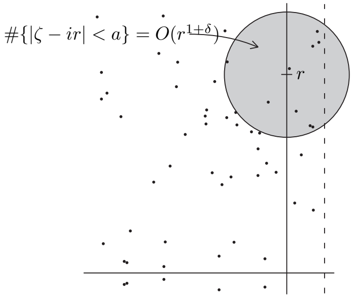
Distribution of Resonances in Strips

Theorem (Guillopé-Zworski) *Suppose that* $X = \mathbb{H}/\Gamma$ *for* Γ *convex co-compact. Then*

$$\#\left\{\zeta \in \mathcal{R}_X : |\zeta| \leq r,\ \Re(\zeta) \geq -M\right\} = \mathcal{O}\left(r^{1+\delta}\right)$$

Note that $1+\delta$ is half the dimension of the trapped set in TX

Datchev-Dyatlov proved a similar bound for resonances near the essential spectrum in asymptotically hyperbolic manifolds.



Spectral Gap (1 of 2)

Theorem (Naud 2010) Suppose Γ is convex co-compact and $\delta < 1/2$. There is an $\varepsilon > 0$ so that

$$\mathcal{R}_X \cap \{\zeta : \delta - \varepsilon \leq \Re(\zeta) \leq \delta\} = \{\delta\}.$$

What is the *spectral gap* between δ and the other resonances of X?

Conjecture (Jakobson-Naud 2011) *There are at most finitely many resonances in the half-plane* $\Re(s) \geq \delta/2 + \varepsilon$.

Theorem (Naud 2012) *Suppose that* $\sigma \geq \delta/2$. *Then*

$$\#\{\zeta \in \mathcal{R}_X : \sigma \leq \Re(s) \leq \delta, \, |Im(s)| \leq r\} = \mathcal{O}\left(T^{1+\delta-\varepsilon(\sigma)}\right)$$

for $\varepsilon(\sigma) > 0$ *as long as* $\sigma > \delta/2$.

Spectral Gap (2 of 2)

Theorem (Naud 2014) *Suppose that* Γ *is convex co-compact. Then*

$$\#\{\zeta \in \mathcal{R}_X : \sigma \leq \Re(s) \leq \delta,\ |Im(s)| \leq T\} < \mathcal{O}\left(T^{1+\tau(\sigma)}\right)$$

Here $\tau(\sigma)$ *satisfies* $\tau(\delta/2) = \delta$, $\tau(\sigma) < \delta$ *for all* $\sigma > \delta/2$, *and* $\tau'(\delta/2) < 0$.