Symmetry factorization of Selberg zeta functions and distribution of resonances

David Borthwick
Emory University
(joint work with Tobias Weich)

CIRM Luminy
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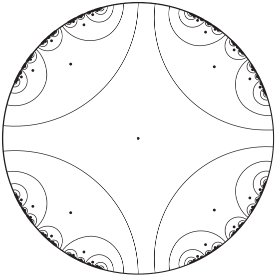
Resonances for hyperbolic surfaces

$(X, g) = \Gamma\backslash\mathbb{H}^2 =$ geometrically finite hyperbolic surface of infinite area.

$\mathcal{R}_X := \Big\{$ poles of $R_X(s) := (\Delta_X - s(1-s))^{-1}$, counted with multiplicity$\Big\}$.

The first resonance occurs at $s = \delta := \dim \Lambda(\Gamma)$. [Patterson: 1976, 1989]



There is a gap $\varepsilon > 0$ such that δ is the only resonance with $\operatorname{Re} s > \delta - \varepsilon$.

[Naud: 2005]

Resonance distribution conjectures

1. Fractal Weyl law [Sjostrand 1990, Lu-Sridhar-Zworski 2003]

$$\#\Big\{\zeta \in \mathcal{R}_X : \operatorname{Re}\zeta \geq \sigma, |\operatorname{Im}\zeta - T| \leq 1\Big\} \asymp T^{\delta} \qquad \text{for some } \sigma < \delta.$$

2. Essential spectral gap [Jakobson-Naud 2012]

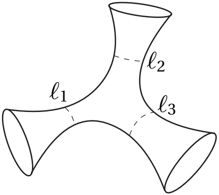
$$\inf\Big\{\sigma : \mathcal{R}_X \cap \{\operatorname{Re} s > \sigma\} \text{ is finite}\Big\} = \frac{\delta}{2}.$$

3. Concentration of decay rates (quantum vs. classical)

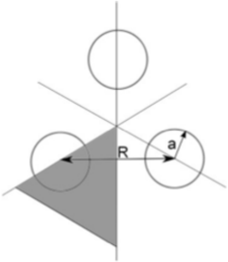
$$\operatorname{Re}(\mathcal{R}_X) \text{ concentrates at } \frac{\delta}{2}.$$

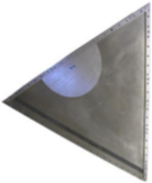
Computations for 3-funnel surfaces

$X(\ell_1, \ell_2, \ell_3)$ = hyperbolic pair of pants + funnel ends.

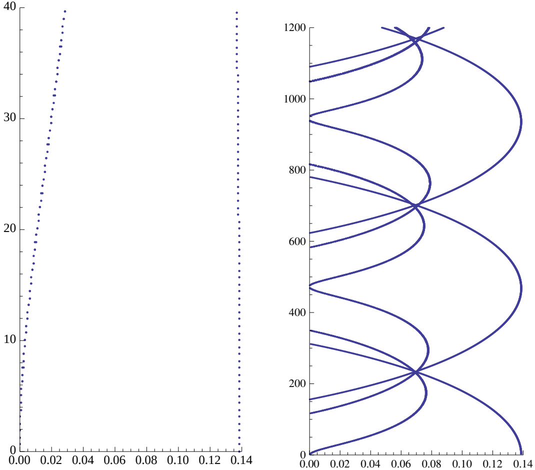


Strong analogies to 3-disk scattering systems

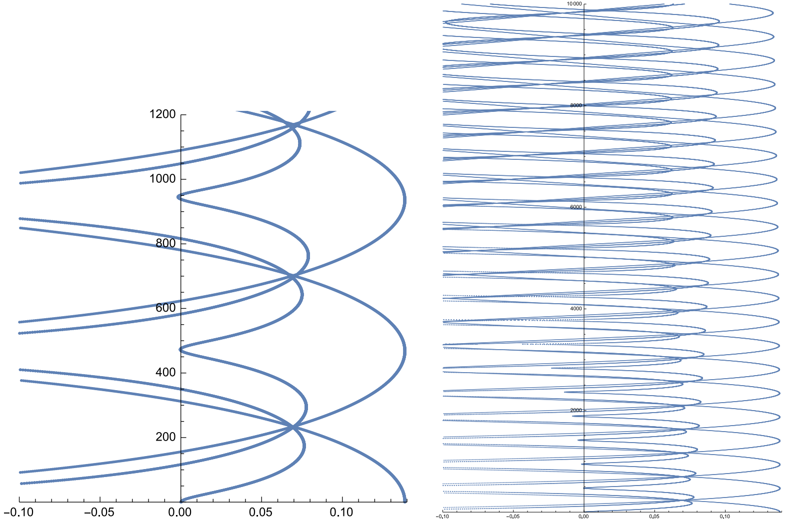




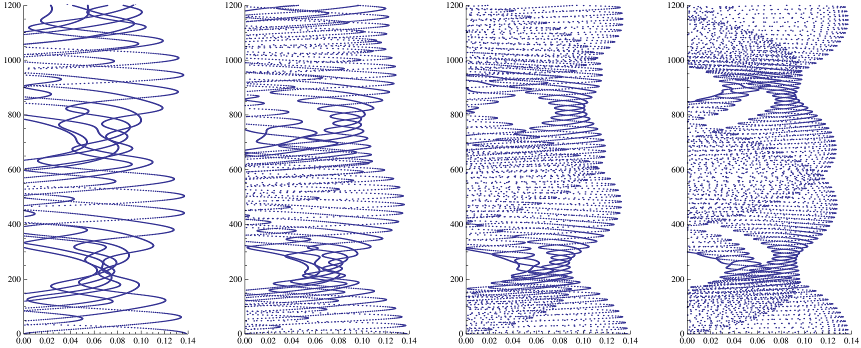
Resonance plots for $X(10, 10, 10)$



Resonance plots for $X(10, 10, 10)$



Resonance chains

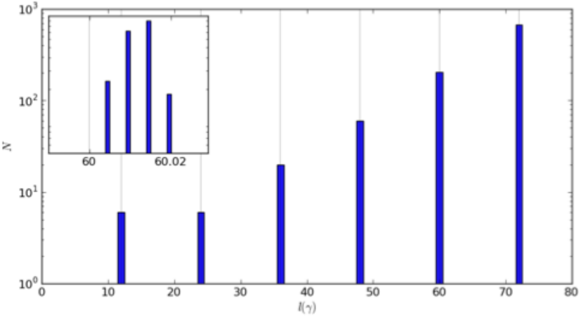


$X(10,10,10.1)$ $X(10,10,10.2)$ $X(10,10,10.3)$ $X(10,10,10.4)$

Similar phenomena have been observed for symmetric n-disk scattering systems and open billiards, both experimentally and numerically.

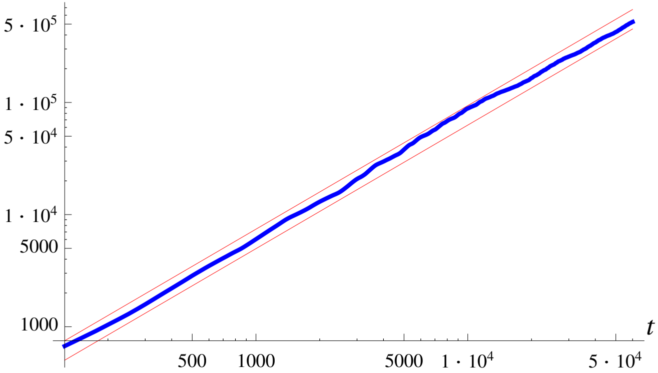
Conjecture Resonance chains are associated to (approximate) clustering of $\mathcal{L}_X$ on $\ell \cdot \mathbb{N}$. [Barkhofen, Faure, Weich 2014].

Clustering for $X(12, 12, 12)$:



Fractal Weyl law

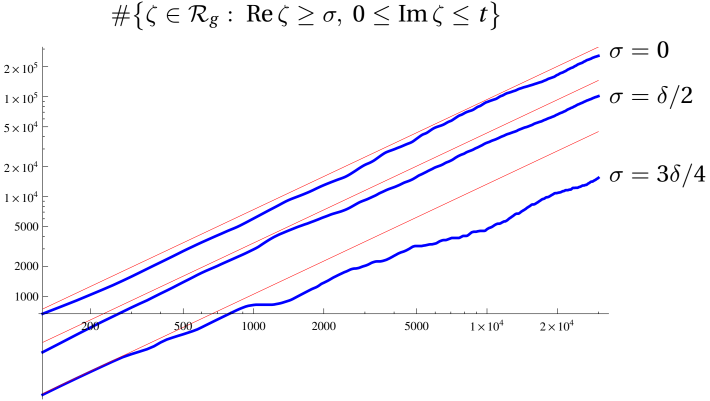
$\#\{\zeta \in \mathcal{R}_X : \text{Re}\,\zeta \geq 0,\ 0 \leq \text{Im}\,\zeta \leq t\}$ for $X(12, 14, 15)$



For $\sigma > \delta/2$, [Naud 2014]

$$\#\{\zeta \in \mathcal{R}_g : \operatorname{Re}\zeta \geq \sigma,\ 0 \leq \operatorname{Im}\zeta \leq t\} = O(t^{1+\tau(\sigma)}),$$

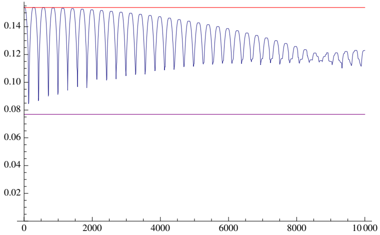
with $\tau(\sigma) < \delta$.



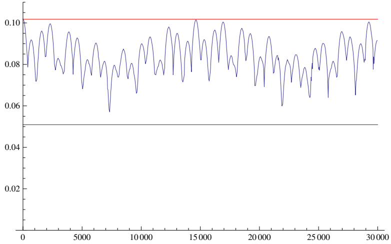
Spectral gap

Envelope function

$$h_w(t) := \max\{\mathrm{Re}\,\zeta : \ \zeta \in \mathcal{R}_X, \ |\mathrm{Im}\,\zeta - t| \le w\}$$



$X(9,9,9)$

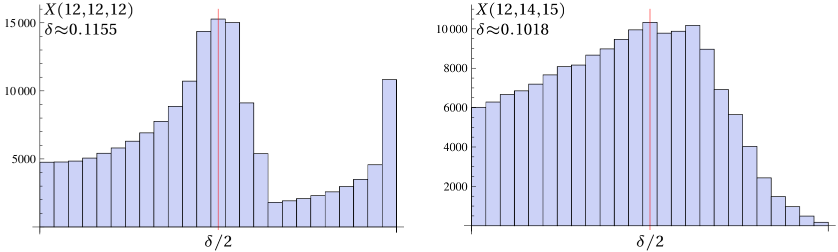


$X(12,14,15)$

Concentration of decay rates

Classical escape rate $= 1 - \delta$.
Quantum decay rate $= \frac{1}{2} - \operatorname{Re}\zeta$ (for a resonance at $s = \zeta$).

$\Longrightarrow$ expect a concentration $\operatorname{Re}\mathcal{R}_X$ at $\frac{\delta}{2}$.



Histograms of $\operatorname{Re}\zeta$ for $0 \operatorname{Im}\zeta \leq 20000$.

Selberg zeta function

For a geometrically finite surface of infinite area, $X = \Gamma\backslash\mathbb{H}$, the Selberg zeta function,

$$Z_X(s) := \prod_{\ell\in\mathcal{L}(\Gamma)} \prod_{k=1}^{\infty} \left(1 - e^{-(s+k)\ell}\right),$$

converges for Re $s > \delta$ and continues meromoprhically.

[Patterson 1989, Guillopé 1992]

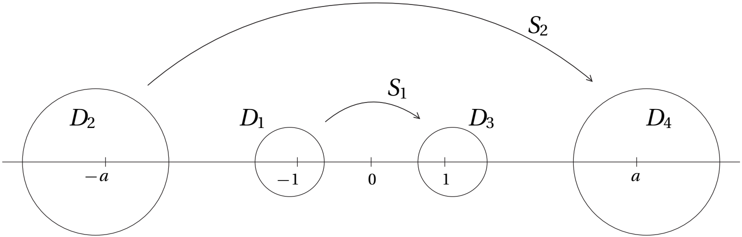
$$\text{Divisor of } Z_X(s) = \begin{cases} \text{zeros at } \mathcal{R}_X \\ \text{topological zeros at } -\mathbb{N}_0 & (\text{multiplicity } \propto \chi(X)) \\ \text{topological poles at } \frac{1}{2} - \mathbb{N}_0 & (\text{multiplicity } \propto n_{\text{cusps}}) \end{cases}$$

[Patterson-Perry 2001, B-Judge-Perry 2005]

No cusps (Γ convex cocompact) $\Longrightarrow Z_X(s)$ is entire.

Zeta function for Schottky groups

[Button 1998]: Γ convex co-compact $\longleftrightarrow$ classical Schottky group.



Γ is generated by $S_1, \ldots, S_r \in PSL(2, \mathbb{R})$, where

$$S_j : \text{interior}(D_j) \to \text{exterior}(D_{j+r}).$$

Dynamical zeta function

Bowen-Series map: On $U := \cup D_j$, define

$$B|_{D_j} := S_j|_{D_j}$$

Ruelle transfer operator $L(s)$ on $L^2_{hol}(U)$:

$$(L(s)u)(z) := \sum_{w \in U:\ Bw = z} B'(w)^{-s} u(w).$$

This setup is called a (holomorphic) iterated function scheme (IFS).

For Γ convex co-compact,

$$Z_X(s) = \det(1 - L(s)).$$

[Pollicott 1991]

Computation of the zeta function

The dynamical realization of the zeta function leads to a “cycle” expansion

$$Z_X(s) = 1 + \sum_{n=1}^{\infty} d_n(s),$$

with an estimate

$$|d_n(s)| \leq Ce^{-c_1 n^2 - c_2 n \operatorname{Re} s + c_3 n |\operatorname{Im} s|}.$$

[Cvitanović-Eckhardt 1989, Jenkinson-Pollicott 2002]

Here $d_n(s)$ defined as a recursive sum over the functions

$$a_n(s) := -\frac{1}{n} \sum_{\sigma \in \mathcal{W}_n} \frac{e^{-s\ell(T_\sigma)}}{1 - e^{-\ell(T_\sigma)}},$$

where

$$\mathcal{W}_n := \left\{ \sigma \in (\mathbb{Z}/2r\mathbb{Z})^n : \ \sigma_{j+1} \neq \sigma_j + r \text{ for } j = 1, \dots, n-1, \text{and } \sigma_1 \neq \sigma_n + r \right\},$$

and $T_\sigma := S_{\sigma_1} \cdots S_{\sigma_n}$.

Symmetry factorization

Theorem [B-Weich 2014] If a holomorphic IFS admits a finite symmetry group G, then the dynamical zeta function factors as a product over $\hat{G}$, the set of irreducible unitary representations of G.

(Complication: the natural action of G on $L^2_{hol}(U)$ is not unitary.)

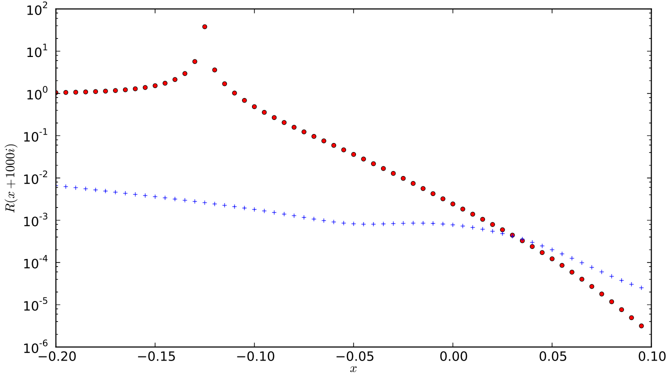
Corollary If a hyperbolic surface has finite symmetry group G, then

$$Z_X(s) = \prod_{\chi \in \hat{G}} Z_X^{\chi}(s).$$

(Complication: the standard Schottky IFS may not reflect the full symmetry group of the surface.)

Symmetry factorization $\rightsquigarrow$ very effective computation of Z_X^χ.

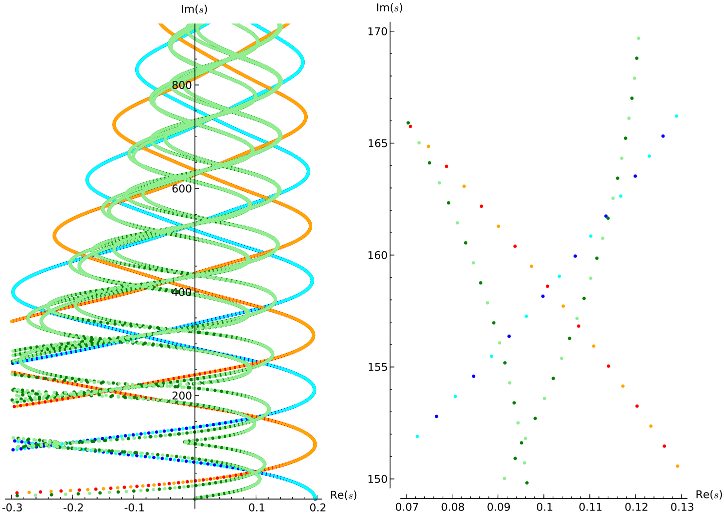
Relative error terms at $s = x + 1000i$:



red = w/o symmetry reduction (using the first 170000 lengths) blue = symmetry reduced (using the first 47 lengths)

Resonances according to representation

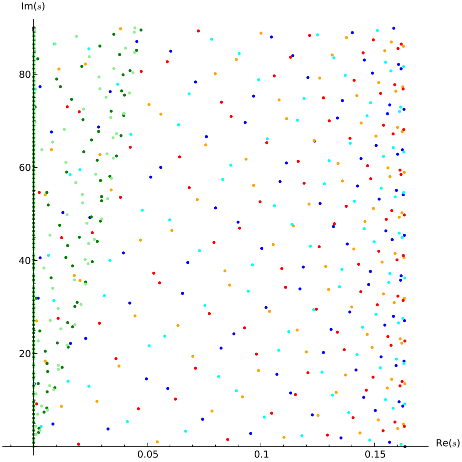
$X(7,7,7) \rightsquigarrow D_3 \times \mathbb{Z}_2$ symmetry



I_1 (dk blue), I_2 (lt blue), II_1 (red), II_2 (orange), III_1 (dk green), III_2 (lt green)

Symmetric four-funnel surfaces

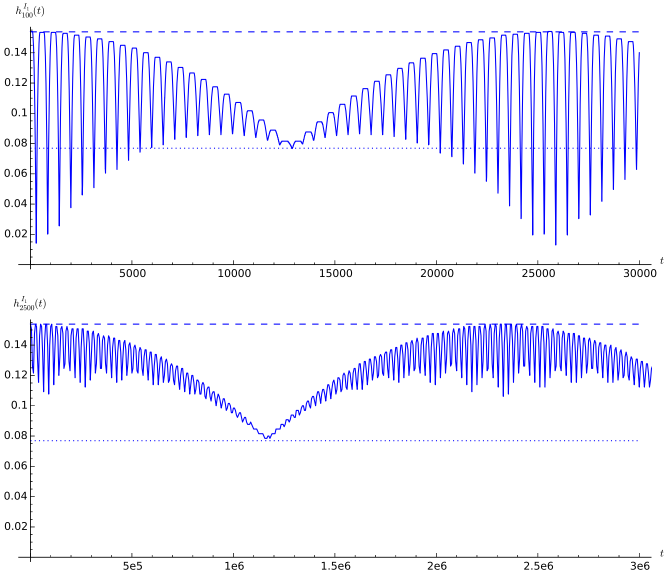
$X(13, 13, 13, 13) \rightsquigarrow D_4 \times \mathbb{Z}_2$

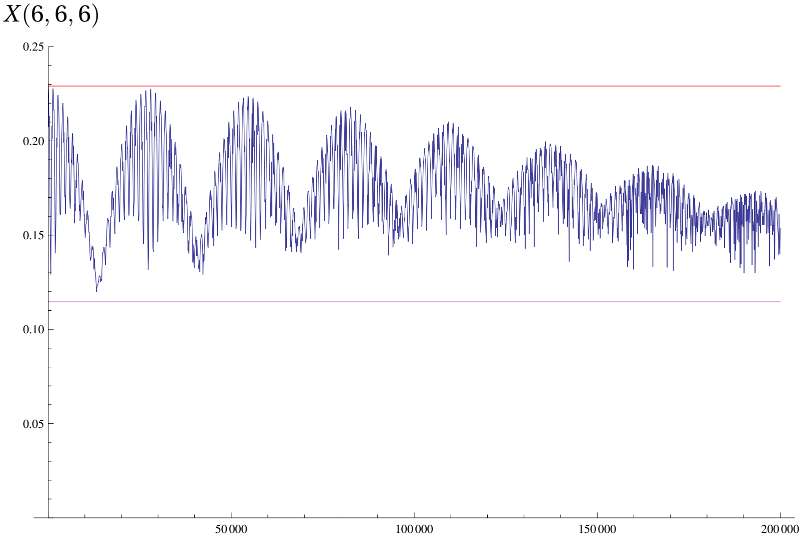


I_1 (dk blue), I_2 (lt blue), II_1 (red), II_2 (orange), V_1 (dark green), V_2 (lt green), III_1, III_2, IV_1, IV_2 (none).

Spectral gap

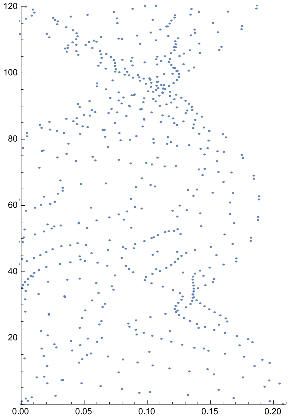
$X(9,9,9)$

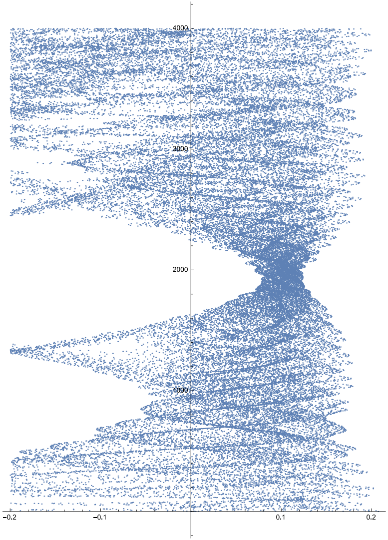



X(6, 6, 6)
0.25
0.20
0.15
0.10
0.05
50 000
100 000
150 000
200 000

D_2 symmetry

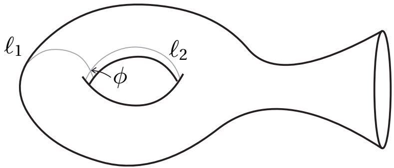
$X(6, 7, 7)$

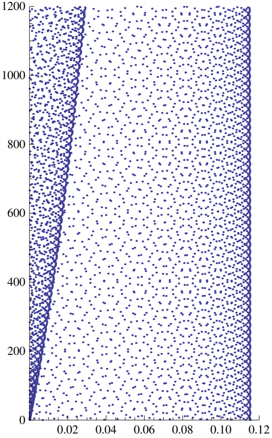




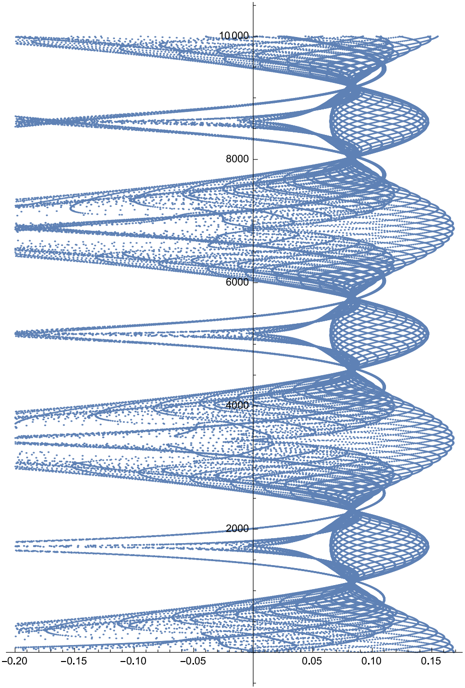
Funneled torus

$Y(\ell_1, \ell_2, \phi)$

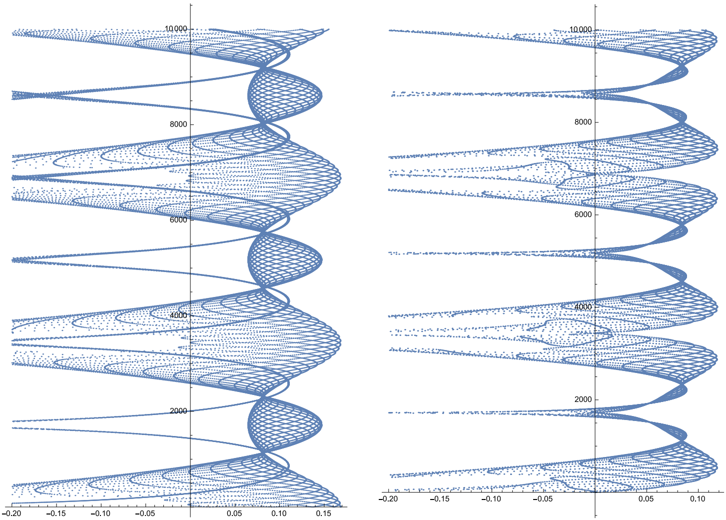




$Y(10,10,\pi/2)$



$Y(7, 7, \pi/2)$


10000
8000
6000
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