A theory of Θ-spherical functions

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Introduction

The theory of spherical functions on symmetric spaces is the result of the interrelation between the analytic theory of special functions and the geometric theory of representations of semisimple Lie groups. The early steps in the function theory on symmetric spaces were made already at the beginning of the last century in the work of Élie Cartan and Hermann Weyl. The spherical functions appeared however only in the 1950s in conjunction with the study of representations of noncompact semisimple Lie groups. Even though it already occurred in the works of Godement and of Gelfand and collaborators, the theory of spherical functions on Riemannian symmetric spaces of the noncompact type is nowadays commonly known as the Harish-Chandra theory. Indeed, Harish-Chandra developed the theory and brought it to a nearly complete form in the late 1950s. According to Harish-Chandra's point of view, the theory of spherical functions is the theory of eigenfunction expansions associated with the invariant differential operators on semisimple symmetric spaces.

In recent years, Harish-Chandra's theory of spherical functions on Riemannian symmetric spaces has been generalized in two different directions. On one side, there have been attempts of extending the theory to non-Riemannian symmetric spaces. A class in which the extension turned out to be quite successful consists of the so-called noncompactly causal (NCC) symmetric spaces. The theory of spherical functions on these spaces started in the middle 1990s with the work of Faraut, Hilgert and Ólafsson. The second generalization is a sort of analytic continuation to a, in general non-geometric, context in which the spherical functions are considered as depending on additional complex parameters (the multiplicities). The Riemannian case is obtained for certain isolated values of these parameters. This is the theory of hypergeometric functions associated with root systems, developed by Heckman and Opdam from the late 1980s. The construction was motivated by the theory of special functions. In one variable, the spherical functions on Riemannian symmetric spaces of rank-one are special cases of the Gaussian hypergeometric function. Heckman and Opdam moved into the opposite direction and constructed hypergeometric functions in more variables by analytically continuing in the multiplicities the spherical functions on Riemannian symmetric spaces of higher rank. In the general context of hypergeometric functions associated with root systems, the geometric intuition is replaced by the full power of the spectral theory. But the geometry can always be regained by specializing the multiplicities. In this way, many interesting and useful properties of the geometrical spherical functions have been deduced by "analytic continuation" of the corresponding properties of the non-geometrical spherical

functions.

In this Habilitationsschrift we present an extension of the theory of Heckman and Opdam which can also contain non-Riemannian situations as particular geometric instances. The construction has been done to include the spherical functions on both Riemannian and NCC symmetric spaces, but some "more room" is left for possible new geometric theories of spherical functions.

The idea of applying the techniques developed by Heckman and Opdam to spherical functions on NCC spaces is not new. Ólafsson has been the first to recognize this possibility when he proved in 1997 the existence of the so-called Harish-Chandra expansions for the spherical functions on NCC symmetric spaces. Ólafsson's expansions for arbitrary multiplicities and some applications of the celebrated shift operators of Opdam also appeared in 1999 in the PhD thesis of Unterberger. These initial attempts have been "parallel" to the theory of Heckman and Odpam, but they could not include it. The first unified definition has been published in [ÓP01a] at the beginning of this year as joint work of Ólafsson and the present author. The motivation of [ÓP01a] was still geometric, more precisely to understand the singularities in the spectral parameter of the spherical functions on the NCC symmetric spaces. The existence of singularities was known from few explicit formulas since the pioneering work of Faraut, Hilgert and Ólafsson. The problem of location of the singularities had however remained unsolved for most of the NCC symmetric spaces, and the Riemannian situation could not suggest any method to attack it, because the spherical functions on Riemannian symmetric spaces are never singular. It was soon recognized that the best way to handle the problem was to use Heckman-Opdam's approach. The construction of the functions now called "Θ-spherical functions" at that point was quite easy.

The definition of the Θ-spherical functions is based on the observation of a specific pattern, which we will outline below. The spherical functions on Riemannian symmetric spaces are functions $\varphi_{\mathrm{RS}}(m;\lambda,a)$, where m is a fixed multiplicity depending on the space, λ is the spectral parameter and a a "radial" space variable. We are employing the subscript RS to indicate that the spherical function $\varphi_{\mathrm{RS}}(m;\lambda,a)$ corresponds to a Riemannian symmetric space. Each function $\varphi_{\mathrm{RS}}(m;\lambda,a)$ is a joint eigenfunction with spectral parameter λ of the commutative algebra of all invariant differential operators on the Riemannian space. For almost all λ's one can explicitly construct a basis for the eigenspace with spectral parameter λ. In fact, it is enough to construct one single eigenfunction, the so-called Harish-Chandra series $\Phi(m;\lambda,a)$, because the action of the Weyl group W on λ yields the remaining elements $\Phi(m;w\lambda,a)$ of the basis. The spherical functions can then be written as linear combinations of the basis elements

$$\varphi_{\mathrm{RS}}(m;\lambda,a)=\sum_{w\in W}c(m;w\lambda)\Phi(m;w\lambda,a), \qquad (*)$$

in which the coefficients are again obtained by the action of W from a single coefficient $c(m;\lambda)$. This representation of the spherical functions in terms of the Harish-Chandra series is commonly known as the Harish-Chandra expansion. The coefficient $c(m;\lambda)$ is the

so-called Harish-Chandra c-function. It governs the asymptotic behavior of the spherical functions for "large" radial variables.

Up to this point, m is fixed by the geometry of the space and can assume only few nonnegative integral values. In their remarkable work, Heckman and Opdam allow the multiplicity m to range arbitrarily (with the symmetry under W as the only restriction). They construct a commutative algebra of differential operators acting on functions of the radial variable a with the property that, for almost all λ, the $\Phi(m;\lambda,a)$ still form a basis for the eigenspace with spectral parameter λ. Moreover, for the values of m which correspond to a Riemannian symmetric space, the algebra agrees with the algebra of invariant differential operators on the symmetric space. Heckman and Opdam use the right-hand side of equality (*) to define spherical functions $\varphi(m;\lambda,a)$ in this non-geometric setting. Finally, they prove that the non-geometric $\varphi(m;\lambda,a)$ possess all regularity properties of their geometric counterparts.

Harish-Chandra expansions also occur in the context of NCC symmetric spaces. As in the Riemannian case, the spherical functions on a NCC symmetric space are common eigenfunctions of the commutative algebra of all invariant differential operators. Every NCC symmetric space has a Riemannian dual with the same radial variables. Restriction to radial variables yields a correspondence between the joint eigenfunctions of the algebras of invariant differential operators on the two spaces. In particular, the spherical functions on NCC symmetric spaces can be written as a linear combination of the basis elements $\Phi(m;w\lambda,a)$ constructed for the Riemannian dual. In 1997, Ólafsson proved that the spherical functions $\varphi_{\mathrm{NCC}}(m;\lambda,a)$ on NCC symmetric spaces admit the expansion

$$\varphi_{\mathrm{NCC}}(m;\lambda,a)=\sum_{w\in W_0} c_{\mathrm{NCC}}(m;w\lambda)\Phi(m;w\lambda,a), \qquad (**)$$

The summation is only over a subset W_0 of the Weyl group W and the coefficient $c_{\mathrm{NCC}}(m;\lambda)$ gives the asymptotic behavior of the function $\varphi_{\mathrm{NCC}}(m;\lambda,a)$ for "large" radial parameters.

The Θ-spherical functions arise very naturally as soon as one recognizes that formulas (*) and (**) are special cases of a more general formula. For this, one needs more information on the structure of the symmetric spaces.

Consider first a Riemannian symmetric space of the noncompact type G/K, where G is a connected noncompact semisimple Lie group with finite center and K is a maximal compact subgroup of G. Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{k}\subset\mathfrak{g}$ be the Lie algebra of K. Then $\mathfrak{k}$ is the eigenspace for the eigenvalue 1 of a nontrivial involution θ on $\mathfrak{g}$, called the Cartan involution. Let $\mathfrak{p}$ denote the eigenspace of θ for the eigenvalue -1. Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, and let $\mathfrak{a}^*$ denote the dual space of $\mathfrak{a}$. The set Σ of (restricted) roots of the pair $(\mathfrak{g},\mathfrak{a})$ consists of all non-zero $\alpha\in\mathfrak{a}^*$ for which

$$\mathfrak{g}^\alpha:=\{X\in\mathfrak{g}:[H,X]=\alpha(H)X \text{ for all } H\in\mathfrak{a}\}$$

contains a nonzero vector. The roots are therefore the common eigenvalues of the commuting family $\{\mathrm{ad}\,H : H\in\mathfrak{a}\}$ of self-adjoint operators on $\mathfrak{g}$. Notice that $\mathfrak{g}^\alpha$ is a real subspace of $\mathfrak{g}$.

Its dimension m_α is the multiplicity of the root α. The multiplicity variable m in (*) denotes the set of multiplicities $\{m_\alpha : \alpha \in \Sigma\}$.

The set of elements $H \in \mathfrak{a}$ for which $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$ is the complement in $\mathfrak{a}$ of a finite union of hyperplanes: each of its connected components is an open convex cone, called a Weyl chamber. We select a Weyl chamber $\mathfrak{a}^+$ as positive, and set $\Sigma^+ := \{\alpha \in \Sigma : \alpha(H) > 0 \text{ for all } H \in \mathfrak{a}^+\}$. The elements of Σ^+ are called the positive roots.

Let $\exp : \mathfrak{g} \to G$ denote the exponential map. Its restriction to $\mathfrak{a}$ induces a diffeomorphism of $\mathfrak{a}$ onto $A := \exp \mathfrak{a}$. We set $A^+ := \exp \mathfrak{a}^+$. The radial "space" variable a of (*) is an element of A^+, whereas the spectral parameter λ lives in the space $\mathfrak{a}_{\mathbb{C}}^*$ of complex-valued linear functionals on $\mathfrak{a}$. The Weyl group W of Σ is the finite group of automorphisms of $\mathfrak{a}^*$ generated by all reflections r_α with $\alpha \in \Sigma$, where

$$r_\alpha(\nu) := \nu - 2\frac{\langle \nu, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha, \qquad \nu \in \mathfrak{a}^*.$$

The action of W on $\mathfrak{a}^*$ extends by $\mathbb{C}$-linearity to an action on $\mathfrak{a}_{\mathbb{C}}^*$ and by duality to an action on $\mathfrak{a}$.

The Harish-Chandra's c-function $c(m;\lambda)$ occurring on the right-hand side of (*) is given by

$$c(m;\lambda) = \kappa \prod_{\alpha \in \Sigma^+} \frac{\Gamma\left(\lambda_\alpha + \frac{m_{\alpha/2}}{4}\right)}{\Gamma\left(\lambda_\alpha + \frac{m_{\alpha/2}}{4} + \frac{m_\alpha}{2}\right)}. \tag{$\dagger$}$$

See also Section 1.4. In (†), we have set

$$\lambda_\alpha := \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}$$

and we convene that $m_\alpha = 0$ whenever $\alpha \notin \Sigma$. The constant κ is chosen to obtain the usual normalization $c(m;\rho) = 1$ when $\rho = 1/2 \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ is the half-sum of the positive roots counted with multiplicities.

When the symmetric space is NCC, there is a special choice for the Euclidean space $\mathfrak{a}$. With this choice, the set Σ of roots of the NCC symmetric space is the set of roots for its dual Riemannian space. But, in this case, Σ has very special additional properties. First of all, it is always reduced (that is $2\alpha \notin \Sigma$ for all $\alpha \in \Sigma$). Moreover, there is a "cone generating" element $Y^0 \in \mathfrak{a}$ inducing in Σ the decomposition $\Sigma = \Sigma_+ \cup \Sigma_0 \cup \Sigma_-$. Here Σ_+ (respectively, Σ_0 and Σ_-) is the set of roots $\alpha \in \Sigma$ satisfying $\alpha(Y^0) > 0$ (respectively, $\alpha(Y^0) = 0$ and $\alpha(Y^0) < 0$). The set Σ_0 is itself a root system. Its Weyl group W_0 is a subgroup of W. It is the summation set occurring in (**).

The c-function $c_{\mathrm{NCC}}(m;\lambda)$ on the right-hand side of (**) is the product

$$c_{\mathrm{NCC}}(m;\lambda) = c_0(m;\lambda) c_\Omega(m;\lambda), \tag{$\ddagger$}$$

where $c_0(m;\lambda)$ is Harish-Chandra's c-function for the root system Σ_0 and

$$c_\Omega(m;\lambda) := \kappa_\Omega \prod_{\alpha \in \Sigma_+} \frac{\Gamma(-\lambda_\alpha - m_\alpha/2 + 1)}{\Gamma(-\lambda_\alpha + 1)}.$$

Also in this case the normalizing constant κ_Ω is chosen so that $c_\Omega(m;\rho) = 1$.

The last piece of structure we need for defining the Θ-spherical functions are the fundamental systems of simple roots. To any set Σ^+ of positive roots in Σ there is a uniquely associated fundamental system Π of simple roots, and the Weyl group W is generated by all reflections $r_j := r_{\alpha_j}$ with $\alpha_j \in \Pi$. When Σ is the root system of a NCC symmetric space, then we can select $\Sigma^+ := \Sigma_0^+ \cup \Sigma_+$, where Σ_0^+ is a choice of positive roots in Σ_0. The fundamental system Π associated with Σ^+ is the union of the fundamental system Π_0 for Σ_0^+ and one simple element from Σ_+. The selection of Π_0 therefore fixes Σ_0, Σ_0^+ and W_0.

Thus, given the Harish-Chandra series $\Phi(m;\lambda,a)$, the right-hand side of (*) and (**) is determined by a subset of Π, namely Π itself in the Riemannian case and Π_0 in the NCC case. This provides the required pattern to define the Θ-spherical functions: Θ is a subset of Π. The elements in Σ which are linear combination of elements of Θ form a root subsystem $\langle\Theta\rangle$ in Σ, and Θ is associated with the positive system $\langle\Theta\rangle^+ := \langle\Theta\rangle \cap \Sigma^+$. The reflections corresponding to elements in Θ generate the parabolic subgroup W_Θ of W. The c-function for Θ will be written as a product

$$c_\Theta(m;\lambda) = c_\Theta^+(m;\lambda)c_\Theta^-(m;\lambda),$$

where c_Θ^+ is the Harish-Chandra c-function for the root system $\langle\Theta\rangle$ and $c_\Theta^-(m;\lambda)$ is modelled after c_Ω. In fact, we need a slight generalization for $c_\Theta^-(m;\lambda)$ since we know that the root systems for NCC spaces are always reduced, while we allow arbitrary root systems. With these ingredients, the Θ-spherical function will be defined as

$$\varphi_\Theta(m;\lambda,a) := \sum_{w\in W_\Theta} c_\Theta(m;\lambda)\Phi(m;\lambda,a)$$

(cf. Definition 1.5.1). The same definition can also be given in the context considered by Heckman and Opdam, and then the hypergeometric functions associated with the root system Σ correspond to the choice $\Theta = \Pi$. In this Habilitationsschrift we shall work in this non-geometric setting and treat the spherical functions on symmetric spaces as (important) special cases.

The central theme of this Habilitationsschrift is the study of the regularity properties of the Θ-spherical functions and the first properties of the associated Θ-spherical transform.

As in the theory of Heckman and Opdam, the Θ-spherical functions are a priori defined only on a subset A^+ of $A = \exp\mathfrak{a}$ and for almost all $\lambda \in \mathfrak{a}_\mathbb{C}^*$. The Riemannian case suggests the possibility of extending the domain of definition to the entire A, and even to a small tubular neighborhood of A in its complexification $A_\mathbb{C}$. This is only possible for the case $\Theta = \Pi$. For arbitrary Θ we shall determine the largest domain $A_\Theta \subset A$ on which all Θ-spherical functions can be extended. Also in this case we can move to a tubular neighborhood in $A_\mathbb{C}$. At the same time, one must deal with the possible singularities in the spectral variable λ. In the case $\Theta = \Pi$, as well as in the Riemannian case, the spherical functions are entire functions of $\lambda \in \mathfrak{a}_\mathbb{C}^*$. For arbitrary Θ, the locus of possible singularities and their order can

be described in terms of the "numerator" of the function c_Θ^-. In the multiplicity variable m there will be no singularities provided we do not follow the normalization of the Θ-spherical functions usually adopted in the Riemannian case. The regularity properties of the Θ-spherical functions are proved in Theorem 1.5.11. For fixed multiplicity functions, this theorem appeared in the joint work [ÓP01a] with G. Ólafsson. The geometrical properties of the set A_Θ are studied in Lemma 1.5.10. As already mentioned, the case $\Theta = \Pi$ corresponds to the absence of singularities. Nevertheless, the tools to determine the location and the nature of the singularities for arbitrary Θ's have been developed by Opdam and Heckman. They have "removed all the singularities" in the case $\Theta = \Pi$. We apply their methods to remove "as many singularities as possible". The interesting part is that we reach a minimal unified description of the singularities. See also Example 1.5.14 below.

For many Θ-spherical functions, the numerator function gives the precise description of the location and order of the singularities in the spectral parameter λ. Usually the singularities are located along a locally-finite infinite family of complex hyperplanes. For very special values of the multiplicity functions it might happen that there are only finitely many singular hyperplanes. The easiest examples are provided by the spherical functions on those NCC symmetric spaces for which all multiplicities are equal to 2. Geometrically, this condition singles out the NCC symmetric spaces G/H with G admitting a complex structure. This is one of the very few cases in which an explicit formula for the spherical function is available. Hence the singularities can be directly determined from the formula. We prove in Theorem 1.6.6 that all Θ-spherical functions corresponding to even multiplicity functions have singularities given by first order poles located along a finite family of complex hyperplanes. The family is explicitly determined. The explicit formula for the spherical functions on Riemannian and NCC symmetric spaces with complex structure is generalized in Theorem 1.6.9 to all Θ-spherical functions corresponding to multiplicities equal to 2. Section 1.6 is based on the joint work [ÓP01b] with G. Ólafsson.

The definition of the spherical transform on Riemannian and NCC symmetric spaces can be easily generalized to define a Θ-spherical Fourier transform. The Θ-spherical Fourier transform of a W_Θ-invariant function f on A_Θ is the W_Θ-invariant function $\mathcal{F}_\Theta f(m)$ on $\mathfrak{a}_{\mathbb{C}}^*$ defined for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ by

$$\mathcal{F}_\Theta f(m;\lambda) := \frac{1}{|W_\Theta|} \int_{A_\Theta} f(a)\, \varphi_\Theta(m;\lambda,a)\mu(m;a)\, da$$

provided the integral converges. Here μ is a suitable density function and da is a normalization of the Haar measure on A. When $\Theta = \Pi$, the transform has been studied by Opdam in [Opd95]. In this case, the convergence of the integral is guaranteed by a sufficiently fast decay at infinity. When $\Theta \neq \Pi$ the situation is more complicated because we also have to take into account the behavior of the integrand along the "walls" of the domain A_Θ. On these walls, the Θ-spherical functions present singularities. The existence of the singularities was known in the case $\Theta = \emptyset$ (when the Θ-spherical functions are essentially the Harish-Chandra series) and also in the geometric case of spherical functions on the NCC symmetric spaces.

The nature and order of the singularities is generally not known. For this reason, all works on the harmonic analysis on NCC symmetric spaces have only considered the spherical transform of functions which are compactly supported in A_Θ, in particular, which stay far away from the walls of A_Θ. This is too restrictive for the harmonic analysis. One would like to work on more structured spaces, such as L^2-spaces or Schwartz spaces.

The central theme of Chapter 2 is to develop estimates which guarantee that the function $\varphi_\Theta(m;\lambda,a)\mu(m;a)$ vanishes along the walls of A_Θ. Removing the problem of the walls of A_Θ, this puts the first stone for a future development of the harmonic analysis of Θ-spherical functions. It is an important first step also for the harmonic analysis on NCC symmetric spaces.

A first group of estimates is treated in Section 2.2, where we estimate the Harish-Chandra series (which are essentially the Θ-spherical functions for $\Theta = \emptyset$). The method employed has its roots in the famous estimates of Gangolli. But various modifications are introduced, and we also employ ideas of Wallach and of Van den Ban and Schlichtkrull. The resulting estimates, stated in Theorem 2.2.18 are quite sharp. They are uniform in $a \in A^+$ and locally uniform in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. There are two important features of these estimates: first, they prove that $\Phi(m;\lambda,a)\mu(m;a)$ vanish on the walls of A^+; second, the domains in the λ-variables on which they are given contain W-invariant domains which cover the entire $\mathfrak{a}_{\mathbb{C}}^*$. The second property allows us to get estimates for $\varphi_\Theta(m;\lambda,a)$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ by means of the estimates for the $\Phi(m;\lambda,a)$ and the defining formula of $\varphi_\Theta(m;\lambda,a)$. The procedure is developed in Section 2.3. The estimates for the Θ-spherical functions are stated in Theorem 2.3.11. The method of estimating $\varphi_\Theta(m;\lambda,a)$ from $\Phi(m;\lambda,a)$ yields estimates which become less sharp with the increasing of Θ. The resulting estimates are generally not sharp. Nevertheless, they are enough to guarantee that $\varphi_\Theta(m;\lambda,a)\mu(m;a)$ vanishes along the walls of A_Θ. See Theorem 2.3.14, Corollary 2.3.15 and Remark 2.3.16.

In the last section, the Θ-spherical transform is inverted on the space of A_Θ-invariant C^∞ functions with compact support in A_Θ. The resulting inversion formula generalizes the formula of Ólafsson for the inversion of the spherical transform on NCC symmetric spaces. In the case of NCC spaces, the inversion has been obtained by reduction to the Riemannian case. We follow the same approach here and reduce the case of the Θ-spherical transform to the inversion for $\Theta = \Pi$, which has been proven by Opdam.

The present work has to be considered as a first step of a much larger project dealing with the harmonic analysis of the Θ-spherical functions. The first motivation to proceed in this direction is the understanding of the harmonic analysis on the NCC symmetric spaces. The L^2 and Schwartz space theory on these spaces have never been studied, and also the Paley-Wiener theorem has been proven only in special cases. We believe that the approach by means of Θ-spherical functions can clarify many of the parts of harmonic analysis which are at this point still obscure. The development of the theory of Θ-spherical functions moves also towards a much more ambitious geometric program, that is to try to understand which spaces can fill the "room" left among the possible choices of Θ. The first natural candidates

are the so-called spaces K_ε of Oshima and Sekiguchi. The dream of a possible unified theory of spherical functions surely justifies this attempt.

Some notational conventions:

1. We adopt the notation commonly used in the theory of symmetric spaces. It differs from the notation employed by Heckman and Opdam in the following ways. The root system R used by Heckman and Opdam is related to our root system Σ by the relation $R = \{2\alpha : \alpha \in \Sigma\}$. Similarly, the multiplicity function k in Heckman-Opdam's work is related to our m by $k_{2\alpha} = m_\alpha/2$. Adopting one of these two notations for the root systems requires also the adoption of the corresponding notation for the multiplicities, if one wishes to keep formulas like $\rho = 1/2 \sum_{\alpha \text{ positive root}} (\text{multiplicity of } \alpha)\alpha$ to hold. A drawback of the notation from symmetric spaces is that there are many factors "2" or "$1/2$" coming out, while they are "absorbed" by the choice of notation in the work of Heckman and Opdam. On the other hand, our initial motivation and final goal is to understand the analysis in a geometrical setting. In the theory of symmetric spaces, the multiplicities are dimensions and hence assume nonnegative integral values, whereas the more efficient notation from the theory by Heckman and Opdam would allow dimensions to be half-integers.

2. We shall often consider functions of several variables in which one or more variables are kept fixed. For such a function, say

$$\begin{aligned} F : A \times B \times C &\to \mathbb{C} \\ (a, b, c) &\mapsto F(a, b, c), \end{aligned}$$

   we shall write $F(a, b)$ to denote the function on C given by $F(a, b)(c) := F(a, b, c)$. Similarly, $F(a)$ will denote the function on $B \times C$ given by $F(a)(b, c) := F(a, b, c)$.

3. $\mathbb{N}$ denotes the set of natural numbers $1, 2, 3, \ldots$;
   $\mathbb{N}_0$ denotes the set of natural numbers with the zero $0, 1, 2, 3, \ldots$;
   $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}$ indicate the real, complex and integer numbers, respectively.

4. $\Re z$ denotes the real part of the complex number z.

5. A list of special symbols occurring in the text is available on p. 98.

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Chapter 1

Θ-spherical functions

1.1 Notation

Let $\mathfrak{a}$ be an l-dimensional real Euclidean vector space with inner product $\langle \cdot, \cdot \rangle$. The real dual $\mathfrak{a}^*$ of $\mathfrak{a}$ consists of all $\mathbb{R}$-linear functionals on $\mathfrak{a}$. The inner product defines an isomorphism of $\mathfrak{a}$ and $\mathfrak{a}^*$ in which the element $\widetilde{H}_\alpha \in \mathfrak{a}$ corresponding to a non-zero $\alpha \in \mathfrak{a}^*$ is determined by the condition that $\alpha(H) = \langle H, \widetilde{H}_\alpha \rangle$ for all $H \in \mathfrak{a}$. The normalized vector $H_\alpha := 2\widetilde{H}_\alpha / \langle \widetilde{H}_\alpha, \widetilde{H}_\alpha \rangle$ satisfies $\alpha(H_\alpha) = 2$. The assignment $\langle \alpha, \beta \rangle := \langle \widetilde{H}_\alpha, \widetilde{H}_\beta \rangle$ defines an inner product in $\mathfrak{a}^*$. For every $\alpha \in \mathfrak{a}^*$, we denote by r_α be the reflection in $\mathfrak{a}^*$ defined by

$$r_\alpha(\lambda) := \lambda - 2\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\, \alpha = \lambda - \lambda(H_\alpha)\alpha, \qquad \lambda \in \mathfrak{a}^*. \tag{1.1}$$

A *root system* in $\mathfrak{a}^*$ is a finite set Σ of non-zero elements which span $\mathfrak{a}^*$ and which satisfy $2\langle \alpha, \beta \rangle / \langle \beta, \beta \rangle \in \mathbb{Z}$ and $r_\alpha(\beta) \in \Sigma$ for all $\alpha, \beta \in \Sigma$. A root system is *reduced* if $2\alpha \notin \Sigma$ for all $\alpha \in \Sigma$. The elements of Σ are called the *roots*. For every $H \in \mathfrak{a}$ satisfying $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$, one can define a set of *positive roots* Σ^+ in Σ by $\Sigma^+ := \{\alpha \in \Sigma : \alpha(H) > 0\}$. A root $\alpha \in \Sigma^+$ is said to be *simple* if it does not decompose as $\alpha = \beta + \gamma$ with β and γ both positive roots. The set Π of all simple roots in Σ^+ is a basis of $\mathfrak{a}^*$ and every element in Σ^+ can be written as a linear combination of elements of Π with nonnegative integral coefficients. Π is called the *fundamental system of simple roots* associated with Σ^+.

Let Σ be a (possibly non-reduced) root system in $\mathfrak{a}^*$. Let Σ^+ be a choice of positive roots in Σ and $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ the fundamental system of simple roots associated with Σ^+. We indicate by $\mathfrak{a}^+$ the open Weyl chamber in $\mathfrak{a}$ on which all elements of Σ^+ are strictly positive. We introduce in $\mathfrak{a}^*$ the lexicographic ordering induced by the basis $\{\alpha_1, \ldots, \alpha_l\}$ and write equivalently $\lambda < \mu$ and $\mu > \lambda$ for $\mu - \lambda > 0$. As usual, the notation $\lambda \leq \mu$ means that either $\lambda < \mu$ or $\mu = \lambda$.

In the following we assume Σ as fixed. We therefore do not include the dependence on Σ in the notation. The *Weyl group* W of Σ is the finite group of orthogonal transformations of $\mathfrak{a}^*$ generated by $\{r_\alpha : \alpha \in \Sigma\}$. We set $r_j := r_{\alpha_j}$. Then $\{r_j : j = 1, \ldots, l\}$ is a minimal set of generators of W.

The complexification $\mathfrak{a}_{\mathbb{C}} := \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathfrak{a}$ can be viewed as the Lie algebra of the complex torus $A_{\mathbb{C}} := \mathfrak{a}_{\mathbb{C}}/\mathbb{Z}\{i\pi H_\alpha : \alpha \in \Sigma\}$. We write $\exp : \mathfrak{a}_{\mathbb{C}} \to A_{\mathbb{C}}$ for the exponential map, with multi-valued inverse $\log$. The split real form $A := \exp(\mathfrak{a})$ of $A_{\mathbb{C}}$ is an abelian subgroup of $A_{\mathbb{C}}$ with Lie algebra $\mathfrak{a}$ such that $\exp : \mathfrak{a} \to A$ is a diffeomorphism. We set $A^+ := \exp \mathfrak{a}^+$. The polar decomposition of $A_{\mathbb{C}}$ is $A_{\mathbb{C}} = AT$, where $T := \exp(i\mathfrak{a})$ is a compact torus in $A_{\mathbb{C}}$ with Lie algebra $i\mathfrak{a}$. Let $\mathfrak{a}_{\mathbb{C}}^*$ be the space of all $\mathbb{C}$-linear functionals on $\mathfrak{a}$. The action of W extends to $\mathfrak{a}$ by duality, to $\mathfrak{a}_{\mathbb{C}}^*$ and $\mathfrak{a}_{\mathbb{C}}$ by $\mathbb{C}$-linearity, and to $A_{\mathbb{C}}$ and A by the exponential map. Moreover, W acts on functions f on any of these spaces by $(wf)(x) := f(w^{-1}x)$, $w \in W$. The $\mathbb{C}$-bilinear extension to $\mathfrak{a}_{\mathbb{C}}^*$ and $\mathfrak{a}_{\mathbb{C}}$ of the inner products $\langle \cdot, \cdot \rangle$ on $\mathfrak{a}^*$ and $\mathfrak{a}$ will be also denoted by $\langle \cdot, \cdot \rangle$.

Definition 1.1.1 A *multiplicity function* on Σ is a W-invariant function $m : \Sigma \to \mathbb{C}$. Setting $m_\alpha := m(\alpha)$ for $\alpha \in \Sigma$, we therefore have $m_{w\alpha} = m_\alpha$ for all $w \in W$. We extend m to $\mathfrak{a}^*$ by putting $m_\alpha = 0$ for $\alpha \notin \Sigma$. We say that a multiplicity function m is *geometric* if there is a Riemannian symmetric space of noncompact type G/K with restricted root system Σ such that m_α is the multiplicity of the root α for all $\alpha \in \Sigma$. Otherwise, m is said to be *non-geometric*. The set $\mathcal{M}$ of all multiplicity functions on Σ is a subspace of the finite-dimensional $\mathbb{C}$-vector space $\mathbb{C}^\Sigma$. The subset of $\mathcal{M}$ consisting of all multiplicity functions m with $m_\alpha \geq 0$ for all $\alpha \in \Sigma$ is denoted by $\mathcal{M}^+$.

The complex dimension $d(\mathcal{M})$ of $\mathcal{M}$ equals the number of W-orbits in Σ. Thus $\mathcal{M}$ can be treated analytically as $\mathbb{C}^{d(\mathcal{M})}$.

The dimension l of $\mathfrak{a}$ will also be called the *(real) rank* of the triple $(\mathfrak{a}, \Sigma, m)$.

Let Θ be an arbitrary subset of the fundamental system $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ associated with Σ^+. The set $\langle \Theta \rangle$ of elements in Σ which can be written as linear combinations of elements from Θ is a subsystem of Σ. Its Weyl group W_Θ is the *parabolic subgroup of* W generated by the reflections $r_j := r_{\alpha_j}$ with $\alpha_j \in \Theta$. Let Σ_{i} and Σ_{u} respectively denote the *indivisible* and *unmultipliable roots* in Σ, i.e.

$$\Sigma_{\mathrm{i}} := \{\alpha \in \Sigma : \alpha/2 \notin \Sigma\}$$
$$\Sigma_{\mathrm{u}} := \{\alpha \in \Sigma : 2\alpha \notin \Sigma\}$$

The indivisible and unmultipliable roots in $\langle \Theta \rangle$ are denoted by $\langle \Theta \rangle_{\mathrm{i}}$ and $\langle \Theta \rangle_{\mathrm{u}}$, respectively. Finally we set $\Sigma^- := -\Sigma^+$ and $\Sigma_{\mathrm{i}}^\pm := \Sigma_{\mathrm{i}} \cap \Sigma^\pm$, $\Sigma_{\mathrm{u}}^\pm := \Sigma_{\mathrm{u}} \cap \Sigma^\pm$, $\langle \Theta \rangle_{\mathrm{i}}^\pm := \langle \Theta \rangle_{\mathrm{i}} \cap \Sigma^\pm$, $\langle \Theta \rangle_{\mathrm{u}}^\pm := \langle \Theta \rangle_{\mathrm{u}} \cap \Sigma^\pm$.

Lemma 1.1.2 *$W_\Theta(\Sigma^\pm \setminus \langle \Theta \rangle^\pm) \subset \Sigma^\pm \setminus \langle \Theta \rangle^\pm$ and $W_\Theta(\Sigma_{\mathrm{i}}^\pm \setminus \langle \Theta \rangle_{\mathrm{i}}^\pm) \subset \Sigma_{\mathrm{i}}^\pm \setminus \langle \Theta \rangle_{\mathrm{i}}^\pm$.*

Proof. For the first inclusion, it is enough to show that $r_i(\Sigma^\pm \setminus \langle \Theta \rangle^\pm) \subset \Sigma^\pm \setminus \langle \Theta \rangle^\pm$ for every generator r_i of W_Θ. As the claim is obvious if $\langle \Theta \rangle^\pm = \Sigma^\pm$, we may assume that $\Sigma^\pm \setminus \langle \Theta \rangle^\pm \neq \emptyset$. If $\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+$, then $\alpha = \sum_{j=1}^l n_j(\alpha)\alpha_j$ with $n_j(\alpha) \in \mathbb{N}_0$ for all j, and there is at least one index j_0 with $\alpha_{j_0} \notin \Theta$ for which $n_{j_0}(\alpha) > 0$. The j_0-th coefficient

of $r_i\alpha = \alpha - \alpha(H_{\alpha_i})\alpha_i$ is also equal to $n_{j_0}(\alpha) > 0$. Hence $r_i\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$. The case of $\Sigma^- \setminus \langle\Theta\rangle^-$ is similar. The second inclusion is a consequence of the first and of the fact that W maps indivisible roots to indivisible roots. □

For $\emptyset \neq \Theta \subset \Pi$ and $m \in \mathcal{M}$ we set

$$\rho_\Theta(m) := \frac{1}{2} \sum_{\alpha \in \langle\Theta\rangle^+} m_\alpha \alpha. \tag{1.2}$$

The abbreviated notation ρ_Θ is used when m is fixed by the context. Moreover, if $\Theta = \Pi$, we write $\rho(m)$ instead of $\rho_\Pi(m)$, i.e.

$$\rho(m) := \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha \in \mathfrak{a}_{\mathbb{C}}^*. \tag{1.3}$$

Lemma 1.1.3 *Suppose $\alpha \in \Theta$. Then $\langle \rho_\Theta(m), \alpha\rangle = (\frac{m_\alpha}{2} + m_{2\alpha})\langle\alpha, \alpha\rangle$.*

Proof. Set $\rho_\alpha := (\frac{m_\alpha}{2} + m_{2\alpha})\alpha$. Then $r_\alpha \in W_\Theta$ maps $\langle\Theta\rangle^+ \setminus \{\alpha, 2\alpha\}$ into itself, so $r_\alpha(\rho_\Theta - \rho_\alpha) = \rho_\Theta - \rho_\alpha$. This implies $\langle \rho_\Theta - \rho_\alpha, \alpha\rangle = 0$, hence the claim. □

For $\alpha \in \Sigma$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we set

$$\lambda_\alpha := \frac{\lambda(H_\alpha)}{2} = \frac{\langle\lambda, \alpha\rangle}{\langle\alpha, \alpha\rangle}. \tag{1.4}$$

Definition 1.1.4 Define

$$P := \{\lambda \in \mathfrak{a}^* : \lambda_\alpha \in \mathbb{Z} \quad \text{for all } \alpha \in \Sigma\},$$
$$P^+ := \{\lambda \in \mathfrak{a}^* : \lambda_\alpha \in \mathbb{N}_0 \quad \text{for all } \alpha \in \Sigma^+\}.$$

We call P the *restricted weight lattice* of Σ. The elements of P^+ are the *dominant restricted weights*.

Observe that $\{2\alpha : \alpha \in \Sigma\} \subset P$. When m is geometric, then P^+ coincides with the set of restrictions to $\mathfrak{a}$ of the highest weights of the finite-dimensional irreducible K-spherical representations of G (see [Hel84], Theorem 4.1, p. 535).

Let $\{\beta_1, \ldots, \beta_l\}$ be the fundamental system of simple roots in Σ_{u}^+ constructed from $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ according to

$$\beta_j := \begin{cases} \alpha_j & \text{if } 2\alpha_j \notin \Sigma \\ 2\alpha_j & \text{if } 2\alpha_j \in \Sigma. \end{cases}$$

For $j = 1, \ldots, l$, define $\mu_j \in \mathfrak{a}^*$ by

$$\frac{\langle\mu_j, \beta_h\rangle}{\langle\beta_h, \beta_h\rangle} = \delta_{j,h}, \qquad j = 1, \ldots, l.$$

As usual, $\delta_{j,h} := 1$ if $j = h$ and 0 otherwise. Then every element of P^+ can be uniquely written as a linear combination of the μ_j's with coefficients in $\mathbb{N}_0$ (see [Hel94], Proposition 4.23, p. 150). The elements μ_j are called the *fundamental restricted weights*.

If $\lambda \in P$, then the exponential $e^\lambda : A_{\mathbb{C}} \to \mathbb{C}$ given by $e^\lambda(h) := e^{\lambda(\log h)}$ is single valued. The e^λ are algebraic characters of $A_{\mathbb{C}}$, i.e. $e^\lambda e^\mu = e^{\lambda+\mu}$ and $e^0 = 1$. In fact, all algebraic characters of $A_{\mathbb{C}}$ are of the form e^λ for some $\lambda \in P$. An *exponential polynomial* is a finite linear combination over $\mathbb{C}$ of exponentials e^λ with $\lambda \in P$. The algebra $\mathbb{C}[A_{\mathbb{C}}]$ of exponential polynomials coincides with the ring of regular functions on $A_{\mathbb{C}}$, when $A_{\mathbb{C}}$ is viewed as an affine algebraic variety. Since P is W-invariant, the Weyl group acts on $\mathbb{C}[A_{\mathbb{C}}]$ according to $w(e^\lambda) := e^{w\lambda}$. The set $A_{\mathbb{C}}^{\mathrm{reg}} := \{h \in A_{\mathbb{C}} : e^{2\alpha(\log h)} \neq 1 \text{ for all } \alpha \in \Sigma\}$ consists of the regular points of $A_{\mathbb{C}}$ for the action of W. Notice that A^+ is a subset of $A_{\mathbb{C}}^{\mathrm{reg}}$. We write $\mathbb{C}[A_{\mathbb{C}}^{\mathrm{reg}}]$ for the algebra of regular functions on $A_{\mathbb{C}}^{\mathrm{reg}}$ and $\mathbb{C}[A_{\mathbb{C}}^{\mathrm{reg}}]^W$ for its W-invariant elements. Notice that

$$\frac{1}{1-e^{-2\alpha}} = 1 - \frac{1}{1-e^{2\alpha}}.$$

Thus $\mathbb{C}[A_{\mathbb{C}}^{\mathrm{reg}}]$ is the subalgebra of the quotient field of $\mathbb{C}[A_{\mathbb{C}}]$ generated by $\mathbb{C}[A_{\mathbb{C}}]$ and $\frac{1}{1-e^{-2\alpha}}$ for $\alpha \in \Sigma^+$.

1.2 The hypergeometric system

Let $\mathrm{S}(\mathfrak{a}_{\mathbb{C}})$ denote the symmetric algebra over $\mathfrak{a}_{\mathbb{C}}$ considered as the space of polynomial functions on $\mathfrak{a}_{\mathbb{C}}^*$, and let $\mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$ denote the subalgebra of invariant elements for the action of the Weyl group W. Every $p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})$ defines a translation invariant differential operators $\partial(p)$ on $A_{\mathbb{C}}$ and on $\mathfrak{a}_{\mathbb{C}}$. We shall also denote by $\mathrm{S}(\mathfrak{a}_{\mathbb{C}})$ the algebra of the differential operators $\partial(p)$ with $p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})$. For $H \in \mathfrak{a}$, the operator $\partial(H)$ is the directional derivative in the direction of H. It satisfies

$$\partial(wH) = w \circ \partial(H) \circ w^{-1} \tag{1.5}$$

for $w \in W$, and $\partial(H)e^\lambda = \lambda(H)e^\lambda$ for $\lambda \in P$.

Let $(\mathfrak{a}, \Sigma, m)$ be a triple as in the previous section. With every $\alpha \in \Sigma^+$ we associate the differential operator

$$\partial_\alpha := \partial(\widetilde{H}_\alpha) = \frac{1}{2}\langle\alpha,\alpha\rangle\partial(H_\alpha). \tag{1.6}$$

Let L_A denote the Laplace operator on A. We define

$$L(m) := L_A + \sum_{\alpha\in\Sigma^+} m_\alpha \coth(\alpha)\,\partial_\alpha, \tag{1.7}$$

where we have set

$$\coth\alpha := \frac{1+e^{-2\alpha}}{1-e^{-2\alpha}}, \qquad \alpha \in \Sigma^+.$$

The differential operator $L(m)$ generalizes to arbitrary multiplicity functions m the radial component $\delta(\omega)$ of the Laplace operator on a Riemannian symmetric space G/K with respect to the left action of K. The differential equation for the spherical functions with spectral parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ which corresponds to $\delta(\omega)$ can therefore be generalized to the case of a triple $(\mathfrak{a}, \Sigma, m)$, also for non-geometric multiplicity functions m, by

$$L(m)\varphi = (\langle \lambda, \lambda \rangle - \langle \rho(m), \rho(m) \rangle)\varphi. \tag{1.8}$$

The problem is to define for arbitrary multiplicity functions the analogue of the entire system of differential equations arising from the radial components of all invariant differential operators on a Riemannian symmetric space. The construction has been carried out by Heckman and Opdam in the series of papers [HO87], [Hec87], [Opd88a] and [Opd88b]. The same authors have successively presented several improvements and simplifications of the original definitions. The resulting system of differential equations, the so-called *hypergeometric system*, can nowadays be constructed by entirely algebraic methods.

The material of this and the following sections is mostly due to Heckman, Opdam and Cherednik. The proofs of the stated results are therefore omitted when easily accessible in the literature.

Let $\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}}) := \mathbb{C}[A_{\mathbb{C}}^{\mathrm{reg}}] \otimes \mathrm{S}(\mathfrak{a}_{\mathbb{C}})$ denote the algebra of differential operators on $A_{\mathbb{C}}$ with coefficients in $\mathbb{C}[A_{\mathbb{C}}^{\mathrm{reg}}]$. The Weyl group W acts on $\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})$ according to

$$w\big(\phi \otimes \partial(p)\big) := w\phi \otimes \partial(wp).$$

We write $\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})^W$ for the subspace of W-invariant elements. The space $\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[W]$ can be endowed with the structure of an associative algebra with respect to the product

$$(D_1 \otimes w_1) \cdot (D_2 \otimes w_2) = D_1 w_1(D_2) \otimes w_1 w_2,$$

where the action of W on differential operators is defined by $(wD)(wf) := w(Df)$ for every sufficiently differentiable function f (see e.g. [Hel84], p. 241). Considering $D \in \mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})$ as element of $\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[W]$, we shall usually write D instead of $D \otimes 1$. The algebra $\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[W]$ is called the algebra of *differential-reflection operators on* $A_{\mathbb{C}}^{\mathrm{reg}}$. It is a left $\mathbb{C}[A_{\mathbb{C}}^{\mathrm{reg}}]$-module. The differential-reflection operators act on functions f on $A_{\mathbb{C}}^{\mathrm{reg}}$ according to $(D \otimes w)f := D(wf)$. It follows that

$$\big((D_1 \otimes w_1) \cdot (D_2 \otimes w_2)\big) f = D_1\big(w_1\big(D_2(w_2 f)\big)\big). \tag{1.9}$$

Define a linear map $\Upsilon : \mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[W] \to \mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})$ by

$$\Upsilon\Big(\sum_j D_j \otimes w_j\Big) := \sum_j D_j.$$

Then $\Upsilon(Q)f = Qf$ for all $Q \in \mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[W]$ and all W-invariant f on $A_{\mathbb{C}}^{\mathrm{reg}}$. Moreover, Υ establishes an algebra homomorphism of

$$(\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[W])^{1 \otimes \mathbb{C}[W]} := \{Q \in \mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[W] : Q \cdot (1 \otimes w) = (1 \otimes w) \cdot Q \text{ for all } w \in W\}$$

into $\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})^W$ (see [HS94], Lemma 1.2.2).

Definition 1.2.1 Let $m \in \mathcal{M}$ and $H \in \mathfrak{a}_{\mathbb{C}}$. The *Dunkl-Cherednik operator* $T(H,m) \in \mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[W]$ is defined by

$$T(H,m) := \partial(H) - \rho(m)(H) + \sum_{\alpha \in \Sigma^+} m_\alpha \alpha(H)(1-e^{-2\alpha})^{-1} \otimes (1-r_\alpha) \tag{1.10}$$

with $\rho(m)$ as in (1.3).

Remark 1.2.2 We have named the Dunkl-Cherednik operators after [Opd00]. These operators appear in the literature also under different names. For instance they are called *trigonometric Dunkl operators* in [Hec97], and *Cherednik operators* in [Opd95]. The name of Dunkl is motivated by his earlier construction of the "Lie algebra version" of the operators treated in Section 1.2.

The Dunkl-Cherednik operators map $\mathbb{C}[A_{\mathbb{C}}]$ into itself. Indeed $\lambda_\alpha \in \mathbb{Z}$ for $\lambda \in P$. Hence the operator

$$\Delta_\alpha := (1-e^{-2\alpha})^{-1} \otimes (1-r_\alpha) \tag{1.11}$$

satisfies

$$\begin{aligned}\Delta_\alpha(e^\lambda) &= \frac{e^\lambda - e^{r_\alpha \lambda}}{1-e^{-2\alpha}} \\ &= \begin{cases} e^\lambda \dfrac{1-e^{-2\lambda_\alpha \alpha}}{1-e^{-2\alpha}} = e^\lambda(1+e^{-2\alpha}+\cdots+e^{-2(m-1)\alpha}) & \text{if } \lambda_\alpha = m \in \mathbb{N} \\ 0 & \text{if } \lambda_\alpha = 0 \\ -e^{r_\alpha\lambda}\dfrac{1-e^{2\lambda_\alpha\alpha}}{1-e^{-2\alpha}} = -e^{r_\alpha\lambda}(1+e^{-2\alpha}+\cdots+e^{-2(|m|-1)\alpha}) & \text{if } \lambda_\alpha = m \in -\mathbb{N}. \end{cases}\end{aligned}$$

The Dunkl-Cherednik operators can also be considered as operators acting on other function spaces, for instance, on the spaces $C^\infty(A)$ and $C_c^\infty(A)$ of C^∞ and compactly supported C^∞ functions on A, or on the analogous spaces of functions on $\mathfrak{a}$. This is always possible because, as can be seen from the Taylor formula, the term $1-r_\alpha$ cancels the apparent singularity on A and $\mathfrak{a}$ arising from the denominator $1-e^{-2\alpha}$.

The most remarkable property of the Dunkl-Cherednik operators is that they commute.

Theorem 1.2.3 *Let $m \in \mathcal{M}$ be fixed. Then the Dunkl-Cherednik operators $\{T(H,m) : H \in \mathfrak{a}_{\mathbb{C}}\}$ form a commuting family of differential-reflection operators in $\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[W]$.*

Proof. This is Theorem 1.2 in [Hec97]. See also Section 2 in [Opd95]. □

Because of Theorem 1.2.3, the map $T_m : \mathfrak{a}_{\mathbb{C}} \to \mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[W]$ defined by

$$T_m(H) := T(H,m), \qquad H \in \mathfrak{a}_{\mathbb{C}},$$

extends uniquely to an algebra homomorphism $T_m : \mathrm{S}(\mathfrak{a}_{\mathbb{C}}) \to \mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[W]$. We set $T(p,m) := T_m(p)$ for $p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})$. Hence for all $p, q \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})$ and $k \in \mathbb{C}$

$$\begin{aligned} T(p+q,m) &= T(p,m) + T(q,m), \\ T(kp,m) &= kT(p,m), \\ T(pq,m) &= T(p,m)T(q,m). \end{aligned}$$

Theorem 1.2.4

(a) *Suppose* $p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$. *Then*

$$T(p,m) \in (\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[W])^{1 \otimes \mathbb{C}[W]}.$$

Setting

$$D(p,m) := \Upsilon\big(T(p,m)\big),$$

we therefore have $D(p,m) \in \mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})^W$.

(b) *The differential operator* $D(p,m)$ *is the unique element of* $\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})^W$ *coinciding on* $\mathbb{C}[A_{\mathbb{C}}^{\mathrm{reg}}]^W$ *with the restriction of* $T(p,m)$.

(c) $\mathbb{D}(\mathfrak{a}, \Sigma, m) := \{D(p,m) : p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W\}$ *is a commutative subalgebra of* $\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})^W$.

Proof. Part (a) is contained in [Opd95], Theorem 2.12 (2). See also [Hec97], pp. 227–228. The proof is based on the relation, due to Cherednik [Che94], between the algebra of Dunkl-Cherednik operators and the degenerate Hecke algebra. See also [Opd95], pp. 79–82. Part (b) follows from (a) and the definition of Υ.

By definition, $D(p,m) = (\Upsilon \circ T_m)(p)$ for all $p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$. Since Υ is an algebra homomorphism on $(\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[W])^{1 \otimes \mathbb{C}[W]}$, part (a) implies that $\Upsilon \circ T_m$ is an algebra homomorphism of $\mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$ onto $\mathbb{D}(\mathfrak{a}, \Sigma, m)$. In particular, for $p, q \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$

$$D(p,m)D(q,m) = D(pq,m),$$

which proves that $\mathbb{D}(\mathfrak{a}, \Sigma, m)$ is commutative. □

Definition 1.2.5 The commutative algebra $\mathbb{D}(\mathfrak{a}, \Sigma, m) := \{D(p,m) : p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W\}$ is called the algebra of *hypergeometric differential operators* associated with the data $(\mathfrak{a}, \Sigma, m)$.

As the name suggests, the elements of $\mathbb{D}(\mathfrak{a}, \Sigma, m)$ are the analogue, for arbitrary multiplicity functions, of the radial parts on A of algebra of invariant differential operators on a Riemannian symmetric space of noncompact type.

Remark 1.2.6 The Dunkl-Cherednik operators have been introduced by Cherednik [Che91] as a modification of so-called *Dunkl-Heckman operators* $S(H,m)$ (see [HS94], Definition 1.1.2, and [Hec97], Remark 1.7). Indeed $T(H,m) = S(H,m) - 1/2 \sum_{\alpha \in \Sigma^+} m_\alpha \alpha(H) r_\alpha$. As well as the Dunkl-Cherednik operators, also the Dunkl-Heckman operators generate the

commutative algebra $\mathbb{D}(\mathfrak{a}, \Sigma, m)$ when restricted to W-invariant functions. However, they do not commute. This is the reason for which Dunkl-Cherednik operators might be preferred. On the other hand, the Dunkl-Heckman operators are W-invariant, i.e. they satisfy the relations $w \circ S(H, m) \circ w^{-1} = S(wH, m)$ for all $H \in \mathfrak{a}_{\mathbb{C}}$ and $w \in W$. The Dunkl-Cherednik operators are not W-invariant, but satisfy the "graded Hecke algebra" relations

$$r_j \circ T(H, m) \circ r_j = T(r_j H, m) - \big(m_{\alpha_j}/2 + m_{2\alpha_j}\big)\alpha_j(H) r_j,$$

i.e.

$$(1 \otimes r_j) \cdot T(H, m) - T(r_j H, m) \cdot (1 \otimes r_j) = \big(m_{\alpha_j}/2 + m_{2\alpha_j}\big)\alpha_j(H)$$

for every $H \in \mathfrak{a}_{\mathbb{C}}$ and every reflection $r_j := r_{\alpha_j}$ associated with a simple root α_j (cf. e.g. [Opd95], p. 79 and Corollary 2.9).

Proposition 1.2.7 *Let $p_L \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$ be the polynomial defined by*

$$p_L(\lambda) := \langle \lambda, \lambda \rangle, \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

Then

$$T(p_L, m) = L_A + \sum_{\alpha \in \Sigma^+} m_\alpha \coth(\alpha)\, \partial_\alpha + \langle \rho(m), \rho(m) \rangle + \tag{1.12}$$
$$+ 2 \sum_{\alpha \in \Sigma^+} m_\alpha \left(\langle \alpha, \alpha \rangle \frac{e^{-2\alpha}}{1 - e^{-2\alpha}} - \langle \alpha, \rho(m) \rangle \right) \Delta_\alpha + \sum_{\alpha, \beta \in \Sigma^+} m_\alpha m_\beta \langle \alpha, \beta \rangle \Delta_\alpha \Delta_\beta.$$

Thus

$$D(p_L, m) = L(m) + \langle \rho(m), \rho(m) \rangle, \tag{1.13}$$

where $L(m)$ is given by (1.7). In particular $L(m) \in \mathbb{D}(\mathfrak{a}, \Sigma, m)$.

Proof. Let $\{H_j\}_{j=1}^l$ be an orthonormal basis of $\mathfrak{a}$. Then for all $\lambda, \mu \in \mathfrak{a}_{\mathbb{C}}^*$

$$\langle \lambda, \mu \rangle = \sum_{j=1}^{l} \lambda(H_j) \mu(H_j). \tag{1.14}$$

Hence $p_L(\lambda) = \sum_{j=1}^l \lambda(H_j)\lambda(H_j) = \big(\sum_{j=1}^l H_j^2\big)(\lambda)$ and

$$T(p_L, m) = T(\sum_{j=1}^{l} H_j^2, m) = \sum_{j=1}^{l} T(H_j, m)^2.$$

Set $\partial_j := \partial(H_j)$ for $j = 1, \ldots, l$. Then

$$\sum_{j=1}^{l} \alpha(H_j) \partial_j = \partial_\alpha.$$

The Laplace operator on A is given in terms of the H_j by $L_A = \sum_{j=1}^l \partial_j^2$. Noticing that

$$2 \sum_{j=1}^{l} \rho(H_j) \partial_j = \sum_{\alpha \in \Sigma^+} m_\alpha \partial_\alpha,$$

we obtain by direct computation

$$\sum_{j=1}^{l} T(H_j,m)^2 = L_A - \sum_{\alpha\in\Sigma^+} m_\alpha\partial_\alpha + \langle\rho(m),\rho(m)\rangle - 2\sum_{\alpha\in\Sigma^+} m_\alpha\langle\alpha,\rho(m)\rangle\Delta_\alpha + \sum_{\alpha,\beta\in\Sigma^+} m_\alpha m_\beta\langle\alpha,\beta\rangle\Delta_\alpha\Delta_\beta + \sum_{\alpha\in\Sigma^+} m_\alpha\left[\partial_\alpha\cdot\Delta_\alpha + \Delta_\alpha\cdot\partial_\alpha\right],$$

where $\cdot$ denotes the product in $\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})\otimes\mathbb{C}[W]$ and Δ_α is as in (1.11). Using (1.9), we have for a differentiable function f

$$\begin{aligned}&\big(\partial_\alpha\cdot\Delta_\alpha + \Delta_\alpha\cdot\partial_\alpha\big)f\\ &\quad= \left[\partial_\alpha\left(1-e^{-2\alpha}\right)^{-1}\right](1-r_\alpha)f + (1-e^{-2\alpha})^{-1}\left\{2\partial_\alpha f - [\partial_\alpha(r_\alpha f) + r_\alpha\partial_\alpha f]\right\}\end{aligned}$$

Because of (1.5),

$$\partial_\alpha(r_\alpha f) = \partial(\widetilde{H}_\alpha)(r_\alpha f) = r_\alpha\partial(r_\alpha\widetilde{H}_\alpha)f = -r_\alpha\partial(\widetilde{H}_\alpha)f = -r_\alpha\partial_\alpha f.$$

Thus

$$\partial_\alpha\cdot\Delta_\alpha + \Delta_\alpha\cdot\partial_\alpha = \partial_\alpha\left(1-e^{-2\alpha}\right)^{-1}\otimes(1-r_\alpha) + 2(1-e^{-2\alpha})^{-1}\partial_\alpha,$$

which yields required expression for $T(p_L,m)$ because

$$\partial_\alpha\left(1-e^{-2\alpha}\right)^{-1}\otimes(1-r_\alpha) = 2\langle\alpha,\alpha\rangle\frac{e^{-2\alpha}}{1-e^{-2\alpha}}\Delta_\alpha.$$

The formula for $D(p_L,m)$ is an immediate consequence of (1.12) because $\Upsilon(\Delta_\alpha) = \Upsilon(\Delta_\alpha\Delta_\beta) = 0$ for all $\alpha,\beta\in\Sigma^+$. □

Remark 1.2.8 Proposition 1.2.7 has been proven in [Hec97] by means of the characterization of $L(m)$ as the unique second order differential operator in $\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})^W$ which has a given symbol and which is symmetric with respect to a certain natural hermitian inner product in $\mathbb{C}[A_{\mathbb{C}}]$. To minimize the amount of structure to be introduced, we have preferred to present a direct computational proof. Our proof has been modeled on Proposition 1.2.3 in [HS94], which proves an analogous statement for the Dunkl-Heckman operators.

For $p\in\mathrm{S}(\mathfrak{a}_{\mathbb{C}})$, let $\widetilde{p}\in\mathrm{S}(\mathfrak{a}_{\mathbb{C}})$ be defined by $\widetilde{p}(\lambda) = p(\lambda-\rho(m))$. Let $\gamma(m):\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})\to\mathrm{S}(\mathfrak{a}_{\mathbb{C}})$ be the algebra homomorphism uniquely determined by

$$\begin{aligned}\gamma(m)\left(\frac{1}{1-e^{-2\alpha}}\right) &:= 1, \qquad \alpha\in\Sigma^+,\\ \gamma(m)\big(\partial(p)\big) &:= \widetilde{p}, \qquad p\in\mathrm{S}(\mathfrak{a}_{\mathbb{C}}).\end{aligned}$$

The *commutant of* $L(m)$ in $\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})^W$ is the algebra

$$\mathbb{D}(m) := \{Q\in\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})^W : L(m)Q = QL(m)\}.$$

Theorem 1.2.9

(a) $\gamma(m)$ *defines an algebra isomorphism of* $\mathbb{D}(m)$ *onto* $\mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$.

(b) $\mathbb{D}(\mathfrak{a},\Sigma,m) \subset \mathbb{D}(m)$ *and* $\gamma(m)\big(D(p,m)\big)(\lambda) = p(\lambda)$ *for all* $p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$.

(c) $\mathbb{D}(\mathfrak{a},\Sigma,m) = \mathbb{D}(m)$ *and* $\gamma(m) : \mathbb{D}(\mathfrak{a},\Sigma,m) \to \mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$, *given by* $\gamma(m)\big(D(p,m)\big)(\lambda) := p(\lambda)$, *is an algebra isomorphism.*

(d) $\mathbb{D}(\mathfrak{a},\Sigma,m)$ *is generated by* $l(=\dim\mathfrak{a})$ *elements.*

Proof. (a) See Theorem 1.3.12 and Remark 1.3.14 in [HS94].

(b) By part (c) of Theorem 1.2.4 and Proposition 1.2.7, $\mathbb{D}(\mathfrak{a},\Sigma,m)$ is a commutative subalgebra of $\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})^W$ and contains $L(m)$. Hence $\mathbb{D}(\mathfrak{a},\Sigma,m) \subset \mathbb{D}(m)$. The equality $\gamma(m)\big(D(p,m)\big)(\lambda) = p(\lambda)$ is part 2 of Theorem 2.12 in [Opd95].

(c) This follows immediately from (a) and (b).

(d) $\mathbb{D}(\mathfrak{a},\Sigma,m)$ is isomorphic to $\mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$, which has precisely l generators according to Chevalley's theorem. □

Because of Proposition 1.2.7, the differential equation (1.8) can be written as

$$D(p_L,m)\varphi = p_L(\lambda)\varphi. \tag{1.15}$$

This observation and Theorem 1.2.9 motivate the following definition.

Definition 1.2.10 The map $\gamma(m) : \mathbb{D}(\mathfrak{a},\Sigma,m) \to \mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$ defined by

$$\gamma(m)\big(D(p,m)\big)(\lambda) := p(\lambda) \tag{1.16}$$

is called the *Harish-Chandra homomorphism*. Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ be fixed. The system of differential equations

$$D(p,m)\varphi = p(\lambda)\varphi, \qquad p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W, \tag{1.17}$$

is called the *hypergeometric system of differential equations with spectral parameter* λ associated with the data $(\mathfrak{a},\Sigma,m)$.

Let $\mu_1,\ldots,\mu_l$ denote the fundamental restricted weights in P^+ (see p. 14). For every $j = 1,\ldots,l$ consider the *orbit sum*

$$z_j := S(e^{\mu_j}) := \sum_{\mu\in W\mu_j} e^{\mu}. \tag{1.18}$$

Chevalley's theorem establishes the algebra isomorphism

$$\mathbb{C}[H]^W \cong \mathbb{C}[z_1,\ldots,z_l]$$

(see [Bou68], pp. 186–188). The commutative algebra $\mathbb{D}(\mathfrak{a},\Sigma,m)$ can therefore be viewed as a system of commuting differential operators on the affine space $W \setminus A_{\mathbb{C}} \cong \mathbb{C}^l$. As a

system of partial differential equations on $W \setminus A_{\mathbb{C}}$, the hypergeometric system has singular points at infinity and along the *discriminant* $\{z \in \mathbb{C}^l : D(z) = 0\}$. Here $D(z) := \Delta(h)^2$ and

$$\Delta(h) = \prod_{\alpha \in \Sigma_{\mathrm{u}}^+} (e^{\alpha} - e^{-\alpha}) \tag{1.19}$$

is the *Weyl denominator*. Notice that $\Delta(h)^2$ is single valued on $A_{\mathbb{C}}$. The subspace $W \setminus A_{\mathbb{C}}^{\mathrm{reg}}$ corresponds to $\mathbb{C}^l \setminus \{z \in \mathbb{C}^l : D(z) = 0\}$.

Remark 1.2.11 A local solution of the system (1.17) near $h \in A_{\mathbb{C}}^{\mathrm{reg}}$ is a $\mathbb{C}$-valued function f with the property that there is an open subset U of $A_{\mathbb{C}}^{\mathrm{reg}}$ containing h such that f is a solution on U of every differential equation of the system. At every point $h \in A_{\mathbb{C}}^{\mathrm{reg}}$, the space $\mathcal{S}(m;\lambda)(h)$ of local solutions of (1.17) near h consists of holomorphic functions and its dimension is the order $|W|$ of the Weyl group (cf. [HS94], Corollary 4.1.8. See also [Opd00], Theorem 6.7). In the language of differential equations, this property is indicated by saying that the hypergeometric system is *holonomic* with *holonomic rank* $|W|$. For "generic" $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, a basis for the solution space of (1.17) on A^+ will be given in Corollary 1.3.8.

Example 1.2.12 (The Euclidean case) When $m \equiv 0$, the root system Σ plays no role. All differential and analytic properties depend only on the Euclidean space $\mathfrak{a}$. We shall therefore write $\mathbb{D}(\mathfrak{a})$ instead of $\mathbb{D}(\mathfrak{a}, \Sigma, 0)$. It consists of the differential operators with constant coefficients. Indeed the Dunkl-Cherednik operators are differential operators and we have $T(p,0) = D(p,0) = \partial(p)$ for all $p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$. The operator $L(0)$ is the Laplacian L_A.

Example 1.2.13 (The rank-one case) The *rank-one case* corresponds to triples $(\mathfrak{a}, \Sigma, m)$ in which $\mathfrak{a}$ is one dimensional. Then the set Σ^+ consists at most of two elements: α and, possibly, 2α. The element $H_\alpha/2 \in \mathfrak{a}$ satisfies $\alpha(H_\alpha/2) = 1$. By setting $H_\alpha/2 \equiv 1$ and $\alpha \equiv 1$, we identify $\mathfrak{a}$ and $\mathfrak{a}^*$ with $\mathbb{R}$, and their complexifications $\mathfrak{a}_{\mathbb{C}}$ and $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}$. The Weyl chamber $\mathfrak{a}^+$ coincides with the half-line $(0, +\infty)$. The Weyl group W reduces to $\{-1, 1\}$ acting on $\mathbb{R}$ and $\mathbb{C}$ by multiplication. Moreover $\rho(m) \equiv m_\alpha/2 + m_{2\alpha}$. We normalize the inner product so that $\langle \alpha, \alpha \rangle = 1$.

The operator $L(m)$ is given on $\mathfrak{a}_{\mathbb{C}}$ by

$$L(m)\varphi = \frac{d^2\varphi}{dz^2} + \big((m_\alpha + m_{2\alpha}) \coth z + m_{2\alpha} \tanh z\big) \frac{d\varphi}{dz}$$

The algebra $\mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$ is generated by the polynomial $p_L(\lambda) = \lambda^2$. Hence $D(p_L, m) = L(m) + \rho^2$ generates $\mathbb{D}(\mathfrak{a}, \Sigma, m)$. The hypergeometric differential system with spectral parameter $\lambda \in \mathbb{C}$ is equivalent to the single 2nd order ODE

$$\frac{d^2\varphi}{dz^2} + \big((m_\alpha + m_{2\alpha}) \coth z + m_{2\alpha} \tanh z\big) \frac{d\varphi}{dz} = (\lambda^2 - \rho^2)\varphi. \tag{1.20}$$

The function $z \mapsto e^z$ maps $\mathfrak{a}_{\mathbb{C}} \equiv \mathbb{C}$ onto $A_{\mathbb{C}} \equiv \mathbb{C}^\times$. Hence $A_{\mathbb{C}}^{\mathrm{reg}} \equiv \mathbb{C} \setminus \{0, \pm 1\}$. The change of variable $\zeta := (1 - \cosh z)/2$ transforms (1.20) into the hypergeometric differential equation

$$\zeta(1-\zeta)\frac{d^2\psi}{d\zeta^2} + [c - (1 + a + b)\zeta]\frac{d\psi}{d\zeta} - ab\zeta = 0 \tag{1.21}$$

with

$$a = \frac{\lambda + \rho}{2}, \qquad b = \frac{-\lambda + \rho}{2}, \qquad c = \frac{m_\alpha + m_{2\alpha} + 1}{2}.$$

Definition 1.2.14 The second-order differential operator

$$S(m) := L_A + \sum_{\alpha \in \Sigma^+} \frac{m_\alpha (2 - m_\alpha - 2m_{2\alpha})}{(e^\alpha - e^{-\alpha})^2} \langle \alpha, \alpha \rangle$$

is called the *Schrödinger operator of the generalized Calogero-Moser system* associated with $(\mathfrak{a}, \Sigma, m)$.

The operator $D(p_L, m) = L(m) + \langle \rho(m), \rho(m) \rangle$ can be transformed by conjugation into the Schrödinger operator. This property will play an important role in the analysis of spherical functions. Set

$$\delta(m) := \prod_{\alpha \in \Sigma^+} (e^\alpha - e^{-\alpha})^{m_\alpha}. \tag{1.22}$$

The function $\delta(m)$ is well defined and single valued on A^+. On $A_{\mathbb{C}}^{\mathrm{reg}}$, it can be viewed as the multivalued holomorphic function obtained from $\delta(m)|_{A^+}$ by analytic continuation.

Theorem 1.2.15 *Let $m \in \mathcal{M}$. Then the following equality of differential operators on $A_{\mathbb{C}}^{\mathrm{reg}}$ holds:*

$$\delta(m/2) \circ [L(m) + \langle \rho(m), \rho(m) \rangle] \circ \delta(-m/2) = L_A + \sum_{\alpha \in \Sigma^+} \frac{m_\alpha (2 - m_\alpha - 2m_{2\alpha})}{(e^\alpha - e^{-\alpha})^2} \langle \alpha, \alpha \rangle. \tag{1.23}$$

Proof. See [HS94], Theorem 2.1.1. □

Remark 1.2.16

(a) Theorem 1.2.15 will play an essential role in the estimates for the Harish-Chandra series in Section 2.2. It is due to Olshanetsky and Perelomov [OP78], who proved it by classification of the root systems. A unified proof appeared in [HO87]. The idea of conjugating the radial part of the Laplace operator by $\delta^{1/2}$ first appeared in the work of Gangolli on spherical functions [Gan71].

(b) Sections 1.2 and 1.3 summarize those parts of the theory of Heckman and Opdam which will be needed in the rest of this paper. The approach via Dunkl-Cherednik operators is only one part of the recent developments of the theory (mainly due to Opdam and to Cherednik). See [Opd00] for further information.

1.3 The Harish-Chandra series

Let $\Lambda := \left\{ \sum_{j=1}^{l} n_j \alpha_j : n_j \in \mathbb{N}_0 \right\}$ be the positive semigroup generated by the fundamental system of simple roots $\Pi := \{\alpha_1, \ldots, \alpha_l\}$ in Σ^+.

As in the classical theory of spherical functions on Riemannian symmetric spaces, the explicit expression of the differential equation (1.8) suggests to look for solutions of the hypergeometric system (1.17) on A^+ with spectral parameter λ that are of the form

$$\Phi(m;\lambda,a) = e^{(\lambda-\rho)(\log a)} \sum_{\mu\in\Lambda} \Gamma_\mu(m;\lambda) e^{-\mu(\log a)}, \qquad a \in A^+.$$

Formally inserting this series into the differential equation (1.8), we obtain the Harish-Chandra type recursion relations

$$\langle \mu, \mu - 2\lambda\rangle \Gamma_\mu(m;\lambda) = 2 \sum_{\alpha\in\Sigma^+} m_\alpha \sum_{\substack{k\in\mathbb{N} \\ \mu-2k\alpha\in\Lambda}} \Gamma_{\mu-2k\alpha}(m;\lambda)\langle \mu+\rho-2k\alpha-\lambda, \alpha\rangle,$$

$$\mu \in \Lambda \setminus \{0\}.$$

for the coefficients $\Gamma_\mu(m;\lambda)$. With the initial condition $\Gamma_0(m;\lambda) = 1$, these relation have unique solutions $\Gamma_\mu(m;\lambda)$ provided $\langle \mu, \mu - 2\lambda\rangle \neq 0$ for all $\mu \in \Lambda \setminus \{0\}$.

Remark 1.3.1 Let $\ell(\mu) := \sum_{j=1}^l n_j$ denote the *level* of $\mu = \sum_{j=1}^l n_j\alpha_j \in \Lambda$. It is easy to check by induction on $\ell(\mu)$ that the recurrence relations imply $\Gamma_\mu(m;\lambda) = 0$ unless $\mu = \sum_{j=1}^l n_j\alpha_j$ with $n_j \geq 0$ and n_j even for all $j = 1,\ldots,l$. Hence the function $\Phi(m;\lambda,a)$ is in fact a sum over $\mu \in 2\Lambda$, that is

$$\Phi(m;\lambda,a) = e^{(\lambda-\rho)(\log a)} \sum_{\mu\in 2\Lambda} \Gamma_\mu(m;\lambda) e^{-\mu(\log a)}, \qquad a \in A^+. \tag{1.24}$$

As in the case for geometric multiplicities (see e.g. [GV88], §4.3, or [Hel84], Chapter IV), the function $\Phi(m;\lambda,a)$ is called the *Harish-Chandra series*.

The Harish-Chandra series is a well-defined formal series for all λ in the set

$$\mathfrak{a}^*_{\mathbb{C},\Lambda} := \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} : \langle \mu, \mu - 2\lambda\rangle \neq 0 \quad \text{for all } \mu \in 2\Lambda\}. \tag{1.25}$$

Proposition 1.3.2 *For every $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Lambda}$, the series*

$$\sum_{\mu\in 2\Lambda} \Gamma_\mu(m;\lambda) e^{-\mu(\log h)} \tag{1.26}$$

converges absolutely and uniformly on compact subsets of $\mathcal{M} \times A^+T$ to a holomorphic function of $(m,h) \in \mathcal{M} \times A^+T$. As a function of λ, it is meromorphic, with at most simple poles located on hyperplanes of the form

$$\mathcal{H}_\mu := \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} : \langle \mu, \mu - 2\lambda\rangle = 0\}$$

for some $\mu \in 2\Lambda$.

For each $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Lambda}$ and for every multiplicity function m, the function $\Phi(m;\lambda)$ is the unique solution of the differential equation (1.8) on A^+ which is of the form

$$e^{\lambda-\rho(m)} + \sum_{\mu\in 2\Lambda\setminus\{0\}} b_\mu e^{-\mu}$$

with coefficients b_μ of at most polynomial growth.

Proof. (Sketch) Let $U \subset \mathcal{M} \times \mathfrak{a}_{\mathbb{C}}^*$ be an open set with compact closure $\overline{U}$, and let $d(m;\lambda)$ be a holomorphic function such that the product $d(m;\lambda)\Gamma_\mu(m;\lambda)$ is holomorphic in a neighborhood of $\overline{U}$ for all $\mu \in 2\Lambda$. Let $H_0 \in \mathfrak{a}^+$ be fixed. Lemma 2.1 in [Opd88b] proves that there is a constant $M = M_{U,H_0}$ such that

$$|d(m;\lambda)\Gamma_\mu(m;\lambda)| \leq Me^{-\mu(H_0)}$$

for all $\mu \in 2\Lambda$ and $(m,\lambda) \in \overline{U}$ (cf. also [HS94], Corollary 4.2.4). This implies that the series

$$\sum_{\mu\in 2\Lambda} d(m;\lambda)\Gamma_\mu(m;\lambda)e^{-\mu(H)}$$

converges absolutely and uniformly on $\overline{U} \times \left[(H_0 + \overline{\mathfrak{a}^+}) \oplus i\mathfrak{a}\right]$. Since H_0 is arbitrary in $\mathfrak{a}^+$, it follows that

$$d(m;\lambda) \sum_{\mu\in 2\Lambda} \Gamma_\mu(m;\lambda)e^{-\mu(H)}$$

is holomorphic in $U \times (\mathfrak{a}^+ \oplus i\mathfrak{a})$. The possible singular hyperplanes come from the left-hand side of the recursions relations for Γ_μ. In fact, $d(m;\lambda)$ can be chosen as the finite product of all polynomial functions $\lambda \mapsto \langle \mu, \mu - 2\lambda\rangle$ intersecting the compact set

$$\{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : (m,\lambda) \in \overline{U} \text{ for some } m \in \mathcal{M}\}.$$

The result can be extended to $A_{\mathbb{C}}$ because for $\mu \in 2\Lambda \subset P$, the function $e^{-\mu(\log h)}$ is single valued and holomorphic on $A_{\mathbb{C}}$.

The proof of the last part of the proposition follows the guidelines of the Riemannian case (see [GV88], pp. 141–144). □

Corollary 1.3.3 *Let $\lambda \in P$ with $\lambda + \rho(m) \in \mathfrak{a}_{\mathbb{C},\Lambda}^*$ be fixed. Then $\Phi(m;\lambda+\rho(m),h)$ defines a holomorphic function of $(m,h) \in \mathcal{M} \times A^+T$.*

Proof. We have

$$\Phi(m;\lambda+\rho(m),h) = e^{\lambda(\log h)} \sum_{\mu\in 2\Lambda} \Gamma_\mu(m;\lambda)e^{-\mu(\log h)}.$$

The series converges to a holomorphic function of $(m,h) \in \mathcal{M} \times A^+T$ by Proposition 1.3.2, and $e^{\lambda(\log h)}$ is holomorphic in $h \in A_{\mathbb{C}}$ because $\lambda \in P$. □

One might expect from Proposition 1.3.2 that the λ-singular set of $\Phi(m;\lambda,h)$ is generally the the union of the hyperplanes $\mathcal{H}_\mu$ with $\mu \in 2\Lambda$. In fact, the λ-singular set is much smaller: all singularities are removable unless $\mu = 2n\alpha$ for some $n \in \mathbb{N}$ and $\alpha \in \Sigma^+$.

Theorem 1.3.4 *There is a connected and simply connected open subset U of T containing the identity element e such that*

$$\begin{aligned}\Phi : \mathcal{M} \times \mathfrak{a}_{\mathbb{C}}^* \times A^+U &\to \mathbb{C}\\ (m,\lambda,h) &\mapsto \Phi(m;\lambda,h)\end{aligned}$$

is a meromorphic function with at most simple poles along the hyperplanes of the form $\mathcal{M} \times \mathcal{H}_{n,\alpha} \times A^+U$*, where*

$$\mathcal{H}_{n,\alpha} := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \lambda_\alpha = n\} \tag{1.27}$$

is a complex hyperplane in $\mathfrak{a}_{\mathbb{C}}^*$ *corresponding to some* $\alpha \in \Sigma_{\mathrm{i}}^+$ *and* $n \in \mathbb{N}$*.*

Proof. This follows from Corollary 2.3 in [Opd88b] together with Proposition 1.3.2. The set U is chosen so that the function $\log$ is single valued on it. Then all functions $e^{(\lambda-\rho)(\log h)}$ ($\lambda \in \mathfrak{a}_{\mathbb{C}}^*$) are single valued and holomorphic on A^+U. □

Remark 1.3.5 We cannot extend Theorem 1.3.4 to the entire A^+T because of the multivalued nature of $\log h$, which forces us to restrict ourselves to a smaller domain A^+U on which all Harish-Chandra series become single valued and holomorphic. This restriction can be removed if we admit multivalued functions, since then we could consider Φ as a meromorphic function on $\mathcal{M} \times \mathfrak{a}_{\mathbb{C}}^* \times A^+T$ which is multivalued in the variable $h \in A^+T$.

Harish-Chandra series have been constructed to be solutions of the differential equation (1.8). Our next step is to describe Opdam's argument to prove that they are indeed solutions of the entire hypergeometric system. Furthermore, exactly as in the Riemannian case, one wishes to use Harish-Chandra series to build a basis of the hypergeometric system with spectral parameter λ. All of this is possible when $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is "generic".

Definition 1.3.6 We say that $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is *generic* if $\lambda_\alpha \notin \mathbb{Z}$ for all $\alpha \in \Sigma$.

Observe that the set of generic elements in $\mathfrak{a}_{\mathbb{C}}^*$ is W-invariant. Indeed, let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ be generic. Then for all $\beta \in \Sigma$

$$(r_\beta \lambda)_\alpha = \lambda_{r_\beta \alpha} \notin \mathbb{Z}$$

since $r_\beta \alpha \in \Sigma$.

For every $p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$, the operator $D(p,m) \in \mathbb{D}(\mathfrak{a}, \Sigma, m)$ possesses the asymptotic expansion

$$D(p,m) = \partial\big(p(\cdot + \rho)\big) + \sum_{\mu \in \Lambda \setminus \{0\}} e^{-\mu} \partial(p_\mu)$$

on A^+, where $p_\mu \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})$ has lower degree than p (see [Opd00], Lemma 6.4). Proposition 1.3.2 uniquely characterizes the $\Phi(m;\lambda,a)$ as solution of the differential equation (1.8). Since the algebra $\mathbb{D}(\mathfrak{a}, \Sigma, m)$ is commutative, the classical argument of Helgason (see e.g. [GV88], p. 144) can be applied to the functions $\Phi(m;\lambda,a)$ to prove the following theorem (see [Opd00], p. 32).

Theorem 1.3.7 *For every generic* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$*, the function* $\Phi(m;\lambda,a)$ *is a solution on* A^+ *of the hypergeometric system (1.17) with spectral parameter* λ*.* □

Corollary 1.3.8 *Let* U *be the open subset of* T *from Theorem 1.3.4. If* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ *is generic, then the set* $\{\Phi(m; w\lambda, a) : w \in W\}$ *is a basis of the solution space on* A^+U *of the hypergeometric system (1.17) with spectral parameter* λ*.*

Proof. If $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is generic, then the series $\Phi(m; w\lambda, h)$ is convergent by Theorem 1.3.4. It is a solution of the entire system of differential equations (1.17) because so is $\Phi(m; \lambda, h)$ and the dependence on λ is on the polynomials $p(\lambda) \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$. The leading exponents $w\lambda - \rho$ ($w \in W$) are distinct for λ generic, hence the set $\{\Phi(m; w\lambda, h) : w \in W\}$ is linearly independent. The claim follows, because the local solution space at every point in $A_{\mathbb{C}}^{\mathrm{reg}}$ has dimension $|W|$ (see Remark 1.2.11). □

Example 1.3.9 (The Euclidean case) In this case $\rho(0) = 0$ and we have plainly $\Phi(0; \lambda, h) = e^{\lambda(\log h)}$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^* \equiv \mathbb{C}^l$.

Example 1.3.10 (The rank-one case) Consider the differential equation on $(0, +\infty)$

$$\frac{d^2\phi}{dt^2} + [(m_\alpha + m_{2\alpha}) \coth t + m_{2\alpha} \tanh t] \frac{d\phi}{dt} = (\lambda^2 - \rho^2)\phi. \tag{1.28}$$

The solution $\Phi(m; \lambda, t)$ behaving asymptotically as $e^{(\lambda-\rho)t}$ for $t \to +\infty$ is

$$\begin{aligned}\Phi(m; \lambda, t) &= (2\cosh t)^{\lambda-\rho}\, {}_2F_1\left(\frac{\rho-\lambda}{2}, \frac{m_\alpha/2 + 1 - \lambda}{2}; 1 - \lambda; \cosh^{-2} t\right) \\ &= (2\sinh t)^{\lambda-\rho}\, {}_2F_1\left(\frac{\rho-\lambda}{2}, \frac{-m_\alpha/2 + 1 - \lambda}{2}; 1 - \lambda; -\sinh^{-2} t\right), \end{aligned} \tag{1.29}$$

where ${}_2F_1$ denotes the *Gaussian hypergeometric function*. The second equality in (1.29) follows from the identity

$${}_2F_1\,(a, b; c; z) = (1 - z)^{-a}\, {}_2F_1\,(a, c - b; c; z/(z - 1))$$

(cf. [E$^+$53], 2.1.(22)). The function $\Phi(m; \lambda, t)$ coincides with the *Jacobi function of second type* $\Phi_\nu^{(a,b)}(t)$ with parameters $a = (m_\alpha + m_{2\alpha} - 1)/2$, $b = (m_{2\alpha} - 1)/2$ and $\nu = -i\lambda$ (see [Koo84], Section 2).

The geometric case of $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$ corresponds to $m_\alpha = 1$ (hence $\rho = 1/2$). In this case

$$\Phi(1; \lambda, t) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(-\lambda + 1)}{\Gamma(-\lambda + 1/2)} Q_{-\lambda-1/2}(\cosh t), \qquad t \in (0, +\infty),$$

where $Q_{-\lambda-1/2}$ is the *Legendre function of second kind* with parameters $\nu = -\lambda - 1/2$ and $\mu = 0$ (cf. [E$^+$53], 3.2(5)). Since we are essentially working on the Lie algebra level, these are also the Harish-Chandra series for the hyperboloid $\mathrm{SO}_0(2, 1)/\mathrm{SO}(2)$, which is locally isomorphic to $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$.

For the hyperboloid $\mathrm{SO}_0(3, 1)/\mathrm{SO}(3) \cong \mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2)$, we have $m_\alpha = 2$. The Harish-Chandra series reduces in this case to

$$\Phi(2, \lambda, t) = \frac{e^{\lambda t}}{2\sinh t}, \qquad t \in (0, +\infty).$$

The Harish-Chandra series is therefore not far from being an exponential function. This is a common feature of all triples $(\mathfrak{a}, \Sigma, m)$ with Σ reduced and $m_\alpha = 2$ for all α (see next example).

Example 1.3.11 (The complex case) Let m be geometric multiplicity of a reduced system Σ. If $m_\alpha = 2$ for all $\alpha \in \Sigma$, then m corresponds to a Riemannian symmetric space of the noncompact type which is endowed with a complex structure. The triple $(\mathfrak{a}, \Sigma, m)$ will be said to correspond to a *complex case*. The constant multiplicity function with all values equal to $k \in \mathbb{C}$ on Σ will be indicated by the number k. In the complex case, $\rho(2) = \sum_{\alpha\in\Sigma^+} \alpha$ and

$$\Delta := \delta(1) = \prod_{\alpha\in\Sigma^+} (e^{\alpha} - e^{-\alpha}) \tag{1.30}$$

is the Weyl denominator (cf. (1.19)). Formula (1.23) becomes

$$\Delta \circ [L(2) + \langle \rho(2), \rho(2)\rangle] \circ \Delta^{-1} = L_A. \tag{1.31}$$

The function on A^+ given by $\Phi(a) := \Delta(a)^{-1} e^{\lambda(\log a)}$ admits the expansion

$$\Phi(a) = e^{(\lambda-\rho)(\log a)} \sum_{\mu\in\Lambda} b_\mu e^{-\mu(\log a)}, \qquad b_0 = 1.$$

It satisfies

$$L(2)\Phi = (\langle \lambda, \lambda\rangle - \langle \rho(2), \rho(2)\rangle)\Phi$$

because of (1.31). Thus

$$\Phi(2; \lambda, a) = \Phi(a) = \Delta(a)^{-1} e^{\lambda(\log a)}. \tag{1.32}$$

1.4 *c*-Functions

Recall that $\lambda_\alpha := \frac{\lambda(H_\alpha)}{2}$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $\alpha \in \Sigma$. Observe that, if $k\alpha \in \Sigma$, then $\widetilde{H}_{k\alpha} = k\widetilde{H}_\alpha$. Hence $H_{k\alpha} = H_\alpha/k$ and $\lambda_{k\alpha} = \lambda_\alpha/k$.

Definition 1.4.1 For a multiplicity function m on Σ, a root $\alpha \in \Sigma^+$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we set

$$c_\alpha^+(m;\lambda) := \frac{\Gamma\left(\lambda_\alpha + \frac{m_{\alpha/2}}{4}\right)}{\Gamma\left(\lambda_\alpha + \frac{m_{\alpha/2}}{4} + \frac{m_\alpha}{2}\right)},$$

$$c_\alpha^-(m;\lambda) := \frac{\Gamma\left(-\lambda_\alpha - \frac{m_{\alpha/2}}{4} - \frac{m_\alpha}{2} + 1\right)}{\Gamma\left(-\lambda_\alpha - \frac{m_{\alpha/2}}{4} + 1\right)}.$$

For any subset $\Theta \subset \Pi$ we define the functions $c_\Theta^\pm$ and $c_\Theta^{\pm,c}$ by

$$c_\Theta^+(m;\lambda) := \prod_{\alpha\in\langle\Theta\rangle^+} c_\alpha^+(m;\lambda),$$

$$c_\Theta^{+,c}(m;\lambda) := \prod_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} c_\alpha^+(m;\lambda),$$

$$c_\Theta^-(m;\lambda) := \prod_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} c_\alpha^-(m;\lambda),$$

$$c_\Theta^{-,c}(m;\lambda) := \prod_{\alpha\in\langle\Theta\rangle^+} c_\alpha^-(m;\lambda).$$

We set $c^+_\emptyset \equiv c^{+,c}_\Pi := 1$ and $c^-_\Pi \equiv c^{-,c}_\emptyset := 1$.

Remark 1.4.2

(a) The interchange of $\langle\Theta\rangle^+$ and $\Sigma^+ \setminus \langle\Theta\rangle^+$ in the formulas for c^+_Θ and c^-_Θ is motivated by the definition of the Θ-spherical functions in Section 1.5.

(b) By definition,

$$c^\pm_\Pi(m;\lambda) = c^\pm_\Theta(m;\lambda)c^{\pm,c}_\Theta(m;\lambda). \tag{1.33}$$

In particular,

$$c^+_\Pi = c^{+,c}_\emptyset \quad \text{and} \quad c^-_\Pi = c^-_\emptyset .$$

(c) The duplication formula for the gamma function

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2) \tag{1.34}$$

gives

$$\frac{1}{\Gamma\Big(\lambda_\alpha + \frac{m_\alpha}{2}\Big)} = \frac{\sqrt{\pi}\;2^{-(\lambda_\alpha+m_\alpha/2)+1}}{\Gamma\Big(\frac{\lambda_\alpha}{2}+\frac{m_\alpha}{4}\Big)\Gamma\Big(\frac{\lambda_\alpha}{2}+\frac{m_\alpha}{4}+\frac{1}{2}\Big)}.$$

From $\lambda_{2\alpha} = \lambda_\alpha/2$, we obtain

$$\begin{aligned} c^+_\alpha(m;\lambda)c^+_{2\alpha}(m;\lambda) &= \frac{\Gamma(\lambda_\alpha)}{\Gamma\Big(\lambda_\alpha+\frac{m_\alpha}{2}\Big)}\,\frac{\Gamma\Big(\frac{\lambda_\alpha}{2}+\frac{m_\alpha}{4}\Big)}{\Gamma\Big(\frac{\lambda_\alpha}{2}+\frac{m_\alpha}{4}+\frac{m_{2\alpha}}{2}\Big)} \\ &= \sqrt{\pi}\,\frac{2^{-(\lambda_\alpha+m_\alpha/2)+1}\;\Gamma(\lambda_\alpha)}{\Gamma\Big(\frac{\lambda_\alpha}{2}+\frac{m_\alpha}{4}+\frac{1}{2}\Big)\Gamma\Big(\frac{\lambda_\alpha}{2}+\frac{m_\alpha}{4}+\frac{m_{2\alpha}}{2}\Big)}. \end{aligned} \tag{1.35}$$

Similarly,

$$c^-_\alpha(m;\lambda)c^-_{2\alpha}(m;\lambda) = \frac{1}{\sqrt{\pi}}\,\frac{\Gamma\Big(-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}+\frac{1}{2}\Big)\Gamma\Big(-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}-\frac{m_{2\alpha}}{2}+1\Big)}{2^{\lambda_\alpha+m_\alpha/2}\;\Gamma(-\lambda_\alpha+1)}. \tag{1.36}$$

Formulas (1.35) and (1.36) yield

$$c^+_\Theta(m;\lambda) = \prod_{\alpha\in\langle\Theta\rangle^+_{\mathrm{i}}} \sqrt{\pi}\,\frac{2^{-(\lambda_\alpha+m_\alpha/2)+1}\;\Gamma(\lambda_\alpha)}{\Gamma\Big(\frac{\lambda_\alpha}{2}+\frac{m_\alpha}{4}+\frac{1}{2}\Big)\Gamma\Big(\frac{\lambda_\alpha}{2}+\frac{m_\alpha}{4}+\frac{m_{2\alpha}}{2}\Big)} \tag{1.37}$$

$$c^-_\Theta(m;\lambda) = \prod_{\alpha\in\Sigma^+_{\mathrm{i}}\setminus\langle\Theta\rangle^+_{\mathrm{i}}} \frac{1}{\sqrt{\pi}}\,\frac{\Gamma\Big(-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}+\frac{1}{2}\Big)\Gamma\Big(-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}-\frac{m_{2\alpha}}{2}+1\Big)}{2^{\lambda_\alpha+m_\alpha/2}\;\Gamma(-\lambda_\alpha+1)}. \tag{1.38}$$

For a fixed geometric m one can recognize that (1.37) is a constant multiple of the classical formula of the Harish-Chandra c-function for the root system $\langle\Theta\rangle$ (see e.g. [GV88], Theorem 4.7.5). Definition 1.4.1 gives a concise representation of the functions $c^\pm_\Theta$. Nevertheless in formulas (1.37) and (1.38) the location of possible singularities and zeros of these functions is more intelligible. See Lemma 1.4.9 below.

(d) When $\alpha/2$ is not a root (e.g. when Σ is reduced), then

$$c_\alpha^+(m;\lambda) := \frac{\Gamma(\lambda_\alpha)}{\Gamma(\lambda_\alpha + m_\alpha/2)}$$
$$c_\alpha^-(m;\lambda) := \frac{\Gamma(-\lambda_\alpha - m_\alpha/2 + 1)}{\Gamma(-\lambda_\alpha + 1)}.$$

Suppose furthermore that m_α is an even integer. The functional equation $\Gamma(z+1) = z\Gamma(z)$ implies

$$c_\alpha^+(m;\lambda) = \frac{1}{\lambda_\alpha(\lambda_\alpha+1)\dots(\lambda_\alpha + m_\alpha/2 - 1)},$$
$$c_\alpha^-(m;\lambda) = \frac{1}{(-\lambda_\alpha - m_\alpha/2 + 1)\dots(-\lambda_\alpha - 1)(-\lambda_\alpha)}.$$

Therefore, when Σ is reduced and m_α is even for all α,

$$c_\Theta^+(m;\lambda) = \prod_{\alpha\in\langle\Theta\rangle^+} \prod_{k=0}^{m_\alpha/2-1} \frac{1}{\lambda_\alpha + k}, \tag{1.39}$$

$$c_\Theta^-(m;\lambda) = \prod_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \prod_{k=-m_\alpha/2+1}^{0} \frac{1}{-\lambda_\alpha + k}, \tag{1.40}$$

are inverse of polynomials. These formulas yield

$$c_\Theta^+(2;\lambda) = \Big[\prod_{\alpha\in\langle\Theta\rangle^+} \lambda_\alpha\Big]^{-1}, \tag{1.41}$$

$$c_\Theta^-(2;\lambda) = \Big[\prod_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} (-\lambda_\alpha)\Big]^{-1} \tag{1.42}$$

in the complex case of $m_\alpha = 2$ for all $\alpha \in \Sigma$.

Example 1.4.3

(a) As already observed in Remark 1.4.2 (c), the function

$$c(m;\lambda) := \frac{c_\Pi^+(m;\lambda)}{c_\Pi^+(m;\rho(m))} \tag{1.43}$$

coincides for geometric multiplicities m with the Harish-Chandra c-function for Riemannian symmetric spaces. See also formula (†) in the introduction. Recall that the Harish-Chandra c-function is normalized to have value 1 at $\rho(m)$. This explains the denominator $c_\Pi^+(m;\rho(m))$.

(b) Suppose that m is a geometric multiplicity function corresponding to a NCC symmetric space G/H. Let $\Theta := \Pi_0$ be the fundamental system for the system Σ_0^+ of positive compact roots. Then $\langle\Theta\rangle^+ = \Sigma_0^+$. The functions

$$c_0(m;\lambda) := \frac{c_\Theta^+(m;\lambda)}{c_\Theta^+(m;\rho(m))}$$
$$c_\Omega(m;\lambda) := \frac{c_\Theta^-(m;\lambda)}{c_\Theta^-(m;\rho(m))}$$

are respectively the Harish-Chandra c-function for the root system Σ_0 and the function c_Ω. Thus

$$c_{\mathrm{NCC}}(m;\lambda) := c_0(m;\lambda)c_\Omega(m;\lambda)$$

is the c-function of G/H. See formula (‡) in the introduction, and [KÓ99] for more information.

Lemma 1.4.4 *$c_\Theta^-(m;\lambda)$ and $c_\Theta^{+,c}(m;\lambda)$ are W_Θ-invariant functions of $\lambda \in \mathfrak{a}_\mathbb{C}^*$.*

Proof. By the W-invariance of m and since $(w\lambda)_\alpha = \lambda_{w^{-1}\alpha}$, we have $c_\alpha^-(m;w\lambda) = c_{w^{-1}\alpha}^-(m;\lambda)$. Since $W_\Theta(\Sigma^+ \setminus \langle\Theta\rangle^+) \subset \Sigma^+ \setminus \langle\Theta\rangle^+$ by Lemma 1.1.2, the claim for $c_\Theta^-(m;\lambda)$ follows. The W_Θ-invariance of $c_\Theta^{+,c}(m;\lambda)$ is proven similarly. □

Lemma 1.4.5 *Let $m \in \mathcal{M}$ be fixed.*

(a) *For all $\alpha \in \Sigma^+$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$*

$$c_\alpha^+(m;\lambda) = \frac{\sin\left[\pi\left(\lambda_\alpha + \frac{m_{\alpha/2}}{4} + \frac{m_\alpha}{2}\right)\right]}{\sin\left[\pi\left(\lambda_\alpha + \frac{m_{\alpha/2}}{4}\right)\right]}\, c_\alpha^-(m;\lambda).$$

(b) *Let $\alpha \in \Pi$ and set $k = \begin{cases} 1 & \text{if } 2\alpha \notin \Sigma \\ 2 & \text{if } 2\alpha \in \Sigma \end{cases}$. Suppose $\sigma_\alpha \in \mathfrak{a}_\mathbb{C}^*$ satisfies*

$$\langle \sigma_\alpha, \alpha\rangle = \left(\frac{m_\alpha}{2} + m_{2\alpha}\right)\langle \alpha, \alpha\rangle.$$

Then for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$

$$c_{k\alpha}^+(m; r_\alpha(\lambda+\sigma_\alpha)) = \frac{\sin\left[\pi(\lambda_{k\alpha})\right]}{\sin\left[\pi\left(\lambda_{k\alpha} + \frac{m_{k\alpha}}{2}\right)\right]}\, \frac{\Gamma(\lambda_{k\alpha}+1)}{\Gamma\left(\lambda_{k\alpha} + \frac{m_{k\alpha}}{2} + 1\right)}. \tag{1.44}$$

In particular, (1.44) holds for $\sigma_\alpha := \rho_\Theta(m)$, where $\Theta \subset \Pi$ and $\alpha \in \Theta$.

Proof. Part (a) follows immediately from the formula $\Gamma(z) = \frac{\pi}{\sin(\pi z)}\frac{1}{\Gamma(1-z)}$. To prove part (b), observe first that

$$[r_\alpha(\lambda+\sigma_\alpha)]_{k\alpha} = \frac{[r_\alpha(\lambda+\sigma_\alpha)]_\alpha}{k} = -\frac{(\lambda+\sigma_\alpha)(H_\alpha)}{2k} = -\lambda_{k\alpha} - \frac{m_\alpha + 2m_{2\alpha}}{2k}.$$

The choice of k ensures that

$$\frac{m_\alpha + 2m_{2\alpha}}{2k} = \frac{m_{k\alpha/2}}{4} + \frac{m_{k\alpha}}{2}.$$

Hence

$$[r_\alpha(\lambda+\sigma_\alpha)]_{k\alpha} + \frac{m_{k\alpha/2}}{4} + \frac{m_{k\alpha}}{2} = -\lambda_{k\alpha}.$$

Equation (1.44) follows then from part (a). The last statement follows from Lemma 1.1.3.

□

We now study the singularities of the functions $c_\Theta^\pm$ introduced in this section. Recall the notation $\mathcal{H}_{r,\alpha} := \{\lambda \in \mathfrak{a}_\mathbb{C}^* : \lambda_\alpha = r\}$ from (1.27). All c-functions are products of ratios of gamma functions applied to linear functionals of $\lambda \in \mathfrak{a}_\mathbb{C}^*$. It is well known that the gamma function $\Gamma(z)$ has no zeros in the complex plane and simple poles at the points $z = -n$ with $n \in \mathbb{N}_0$. The Γ-functions occurring in the numerator of the c-functions therefore contribute simple poles along certain hyperplanes $\mathcal{H}_{r,\alpha}$, whereas those in the denominator contributes simple zeros along other hyperplanes $\mathcal{H}_{r,\alpha}$. To analyze the various cancellations which might take place, it is convenient to introduce separate notation for the numerators and the denominators of the c-functions. In particular, formula (1.35) suggests the following definition.

Definition 1.4.6 The function

$$n_\Theta^-(m;\lambda) := \prod_{\alpha \in \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+} \Gamma\Big(-\frac{\lambda_\alpha}{2} - \frac{m_\alpha}{4} + \frac{1}{2}\Big)\Gamma\Big(-\frac{\lambda_\alpha}{2} - \frac{m_\alpha}{4} - \frac{m_{2\alpha}}{2} + 1\Big) \tag{1.45}$$

will be called the *numerator of* $c_\Theta^-(m;\lambda)$.

The function

$$d_\Theta^-(m;\lambda) := \prod_{\alpha \in \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+} \sqrt{\pi}\, 2^{\lambda_\alpha + m_\alpha/2}\, \Gamma(-\lambda_\alpha + 1) \tag{1.46}$$

will be called the *denominator of* $c_\Theta^-(m;\lambda)$.

Remark 1.4.7 Suppose $2\alpha \notin \Sigma^+$ for all $\alpha \in \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+$, which occurs for instance when Σ is reduced. Then we obtain from the duplication formula (1.34)

$$n_\Theta^-(m;\lambda) := \prod_{\alpha \in \Sigma^+ \setminus \Theta^+} \sqrt{\pi}\, 2^{\lambda_\alpha + m_\alpha/2}\, \Gamma\left(-\lambda_\alpha - \frac{m_\alpha}{2} + 1\right). \tag{1.47}$$

Lemma 1.4.8 *Suppose* $\mathcal{H}_{n,\alpha} = \mathcal{H}_{m,\beta}$ *with* $\alpha, \beta \in \Sigma_{\mathrm{i}}^+$ *and* $n, m \in \mathbb{Z}$. *Then* $\alpha = \beta$ *and* $n = m$.

Proof. Let $\lambda_0 \in \mathcal{H}_{n,\alpha}$. Then

$$\{\lambda - \lambda_0 : \lambda \in \mathcal{H}_{n,\alpha}\} = \mathcal{H}_{0,\alpha} = \{\nu \in \mathfrak{a}_\mathbb{C}^* : \langle \nu, \alpha\rangle = 0\}.$$

Since $\mathcal{H}_{n,\alpha} = \mathcal{H}_{m,\beta}$, we deduce

$$\begin{aligned}\{\nu \in \mathfrak{a}_\mathbb{C}^* : \langle \nu, \alpha\rangle = 0\} &= \{\lambda - \lambda_0 : \lambda \in \mathcal{H}_{n,\alpha}\} \\ &= \{\lambda - \lambda_0 : \lambda \in \mathcal{H}_{m,\beta}\} \\ &= \{\nu \in \mathfrak{a}_\mathbb{C}^* : \langle \nu, \beta\rangle = 0\}.\end{aligned}$$

Since $\alpha, \beta \in \mathfrak{a}^*$ and $\langle\cdot,\cdot\rangle$ is an inner product in $\mathfrak{a}^*$, it follows that $\alpha = k\beta$ for some $k \neq 0$. But $\alpha, \beta \in \Sigma_{\mathrm{i}}^+$, whence $\alpha = \beta$. The equality $\mathcal{H}_{n,\alpha} = \mathcal{H}_{m,\alpha}$ then implies $n = m$ because $r\alpha \in \mathcal{H}_{n,\alpha}$ if and only if $r = n$. □

Lemma 1.4.9

(a) *The singularities of the meromorphic function* $c_\Theta^+(m;\lambda)$ *are at most simple poles located along the hyperplanes*

$$\mathcal{H}_{-n,\alpha} \qquad \text{with } \alpha \in \langle\Theta\rangle_{\mathrm{i}}^+ \text{ and } n \in \mathbb{N}_0\,.$$

(b) *The singularities of the numerator* $n_\Theta^-(m;\lambda)$ *are poles located along the hyperplanes*

$$\mathcal{H}_{-\frac{m_\alpha}{2}+(2n-1),\alpha} \qquad \text{with } \alpha \in \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+ \text{ and } n \in \mathbb{N},$$
$$\mathcal{H}_{-\frac{m_\alpha}{2}-m_{2\alpha}+2n,\alpha} \qquad \text{with } \alpha \in \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+ \text{ and } n \in \mathbb{N}.$$

Therefore all poles are simple if and only if $m_{2\alpha}$ *is not an odd integer for all* $\alpha \in \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+$. *If* $m_{2\alpha}$ *is an odd integer for some* $\alpha \in \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+$, *then there are at most double poles.*

In particular, the poles are always simple when $2\alpha \notin \Sigma^+$ *for all* $\alpha \in \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+$ *(e.g. when* Σ *is reduced). In this case the singularities of the numerator* $n_\Theta^-(m;\lambda)$ *are simple poles located along the hyperplanes*

$$\mathcal{H}_{-\frac{m_\alpha}{2}+n,\alpha} \qquad \text{with } \alpha \in \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+ \text{ and } n \in \mathbb{N}.$$

(c) *The singularities of the denominator* $d_\Theta^-(m;\lambda)$ *are simple poles located along the hyperplanes*

$$\mathcal{H}_{n,\alpha} \qquad \text{with } \alpha \in \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+ \text{ and } n \in \mathbb{N}.$$

Proof. The list of singular hyperplanes is obtained from the list of poles of the gamma function. The order of the singularities follows from Lemma 1.4.8. Observe, in particular, that the singularities of n_Θ^- are simple if and only if $-m_\alpha/2+(2n-1) \neq -m_\alpha/2-m_{2\alpha}+2m$ for all $n, m \in \mathbb{N}$, i.e. $m_{2\alpha} \neq 2(m-n)+1$ for all $n, m \in \mathbb{N}$. The list of singularities of n_Θ^- in the case $2\alpha \notin \Sigma^+$ for all $\alpha \in \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+$ is obtained from the formula in Remark 1.4.7. □

Remark 1.4.10 In the geometric case, it is known that, if 2α is a root, then $m_{2\alpha}$ is odd (see e.g. [Hel78], p. 530). In this case, by part (c) of Lemma 1.4.9, all poles of n_Θ^- are simple if and only if 2α is not a root for all $\alpha \in \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+$.

1.5 Θ-spherical functions

Definition 1.5.1 Let $\Theta \subset \Pi$ be any subset of simple roots and let $c_\Theta^\pm$ be the functions introduced in Definition 1.4.1. Let U be the open subset of T from Theorem 1.3.4. The function on A^+U defined for generic $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ by

$$\varphi_\Theta(m;\lambda,h) := c_\Theta^-(m;\lambda) \sum_{w \in W_\Theta} c_\Theta^+(m;w\lambda)\Phi(m;w\lambda,h), \qquad h \in A^+U \tag{1.48}$$

is called the Θ-*spherical function of spectral parameter* λ.

As a linear combination of the Harish-Chandra series $\Phi(m; w\lambda, h)$, the Θ-spherical function of spectral parameter λ is by construction a solution of the hypergeometric system of spectral parameter λ.

The function φ_Θ is a priori defined only for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ generic and $h \in A^+U$. The main result of this section is Theorem 1.5.11. It states that there exists a certain W_Θ-invariant open domain $U_\Theta \subset A_{\mathbb{C}}$ such that the right-hand side of (1.48) extends as a W_Θ-invariant holomorphic function of $h \in U_\Theta$ and as a W_Θ-invariant meromorphic function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Theorem 1.5.11 moreover determines the possible λ-singularities of the extended function. In the fundamental case $\Theta = \Pi$, the theorem is due to Heckman and Opdam (see e.g. [HS94], §4.3–4.4, and references therein). They have been able to prove that $\varphi_\Pi(m; \lambda, h)$ extends as a W-invariant holomorphic function in $\mathcal{M} \times \mathfrak{a}_{\mathbb{C}}^* \times U_\Pi$, where U_Π denotes a W-invariant neighborhood of A in $A_{\mathbb{C}}$. Their results parallel the regularity properties of the spherical functions on Riemannian symmetric spaces.

Example 1.5.2

(a) When $\Theta = \Pi$, then

$$\varphi_\Pi(m; \lambda, h) = \sum_{w \in W} c_\Pi^+(m; w\lambda)\Phi(m; w\lambda, h)$$

has been first considered by Heckman and Opdam in [HO87]. The *hypergeometric function* associated with the root system Σ is

$$F(m; \lambda, h) := \frac{\varphi_\Pi(m; \lambda, h)}{c_\Pi^+(m; \rho(m))} = \sum_{w \in W} c(m; w\lambda)\Phi(m; w\lambda, h),$$

where $c(m; \lambda)$ is Harish-Chandra's c-function of formula (1.43). The chosen normalization for $F(m; \lambda, h)$ is motivated by the requirement $F(m; \lambda, e) = 1$, in analogy to the classical requirement imposed on the spherical functions on Riemannian symmetric spaces. The computation of the value at e is a non-trivial matter which has been carried out by Opdam using the so-called shift operators (see e.g. [HS94], pp.67–68).

For geometric multiplicities m, the hypergeometric functions $F(m; \lambda, h)$ agree with Harish-Chandra's spherical functions. In particular, $\varphi_\Pi(m; \lambda, h)$ reduces to a spherical function up to a multipliciative factor depending only on the multiplicities.

(b) Suppose Σ is the restricted root system of a NCC symmetric space (in particular Σ is reduced). Let Π_0 be the fundamental system for the positive compact roots Σ_0^+ and set $\Theta := \Pi_0$. Then $\langle\Theta\rangle = \Sigma_0$ and $W_\Theta = W_0$. Up to a factor depending only on the multiplicities, the Θ-spherical function reduces to Unterberger's for arbitrary multiplicities, and to the spherical functions on NCC symmetric spaces for geometric multiplicities.

(c) When $\Theta = \emptyset$, then $W_\emptyset = \{\mathrm{id}\}$, hence $\varphi_\emptyset(m; \lambda, h) = c_\emptyset^-(m; \lambda)\Phi(m; \lambda, h)$.

As already mentioned, when $\Theta = \Pi$ the Θ-spherical functions turn out to be entire in every variable. In the case of an arbitrary Θ one cannot expect so much regularity. This can already be observed in the case $\Theta = \emptyset$, when $\varphi_\emptyset(m;\lambda,h) = c_\emptyset^-(m;\lambda)\Phi(m;\lambda,h)$ is a multiple of the Harish-Chandra series. Even in the simple case of $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ (which has been treated in Example 1.3.10 and in part (d) of Remark 1.4.2) we see that the function

$$\varphi_\emptyset(2;\lambda,t) = e^{\lambda t}/(2\lambda\sinh t), \qquad t \in (0,\infty) \equiv A^+,$$

cannot be continuously extended in $t = 0$. Moreover, $\varphi_\emptyset(2;\lambda,t)$ is not entire as a function of $\lambda \in \mathbb{C} \equiv \mathfrak{a}_\mathbb{C}^*$ because of the simple pole at $\lambda = 0$.

Despite the example just presented, the λ-singularities of φ_Θ need not be along finitely many singular hyperplanes. The general rank-one case in Example 1.3.10 shows that, as a function of $\lambda \in \mathfrak{a}_\mathbb{C}^*$, the Harish-Chandra series $\Phi(m;\lambda,t)$ might have a pole at each $\lambda = n \in \mathbb{N}$. This is of course a special case of Theorem 1.3.4, but one can easily deduce it from the explicit formula (1.29). Indeed, ${}_2F_1\,(a,b;c;z)\,/\Gamma(c)$ is a holomorphic function of the parameter $a,b,c \in \mathbb{C}$ when z is fixed and $|z| < 1$ (see e.g. [E$^+$53], 2.1.6). Recalling the identifications introduced in Example 1.3.10 for the rank-one case, we can write from (1.38)

$$\varphi_\emptyset(m;\lambda,t) = n_\emptyset^-(m;\lambda)\,\frac{(2\cosh t)^{\lambda-\rho}}{\sqrt{\pi}\,2^{\lambda+m_\alpha/2}}\,\frac{{}_2F_1\left(\frac{\rho-\lambda}{2},\frac{m_\alpha/2+1-\lambda}{2};1-\lambda;\cosh^{-2}t\right)}{\Gamma(1-\lambda)}, \tag{1.49}$$

where

$$n_\emptyset^-(m;\lambda) = \Gamma\Big(-\frac{\lambda}{2}-\frac{m_\alpha}{4}+\frac{1}{2}\Big)\Gamma\Big(-\frac{\lambda}{2}-\frac{m_\alpha}{4}-\frac{m_{2\alpha}}{2}+1\Big). \tag{1.50}$$

The property of holomorphic dependence of the hypergeometric functions on its parameters mentioned above shows that for every fixed m and every $t \in (0,\infty)$, the function $\varphi_\emptyset(m;\lambda,t)/n_\emptyset^-(m;\lambda)$ is entire in $\lambda \in \mathbb{C} \equiv \mathfrak{a}_\mathbb{C}^*$. The possible singularities of $\varphi_\emptyset(m;\lambda,t)$ are therefore among the singularities of the numerator function $n_\emptyset^-(m;\lambda)$ and with lower or equal order of singularity. As seen for $m = 2$, it is possible that for some values of the multiplicity function m not all poles of $n_\emptyset^-(m;\lambda)$ are also poles of $\varphi_\emptyset(m;\lambda,t)$. On the other hand, when for instance Σ is reduced and $m = 1$, then every pole of $n_\emptyset^-(m;\lambda)$ is simple and occurs as a pole of $\varphi_\emptyset(m;\lambda,t)$ for some $t \in (0,\infty)$ (see Example 1.5.14 below). Thus $n_\emptyset^-(m;\lambda)$ provides us with a "minimal" unified description of the λ-singularities of the functions $\varphi_\emptyset$.

When $\Theta = \emptyset$, the above argument can be easily modified and applies also to arbitrary triples $(\mathfrak{a},\Sigma,m)$ of rank not equal to one. In this case we need Theorem 1.3.4. In fact, we do not even need to restrict ourselves to A^+, but we can work in the domain $A^+U \subset A_\mathbb{C}$ occurring in the definition of φ_Θ.

Proposition 1.5.3 *Suppose $m \in \mathcal{M}$ and $h \in A^+U$ are fixed. Then the function*

$$\frac{\varphi_\emptyset(m;\lambda,h)}{n_\emptyset^-(m;\lambda)}$$

extends as an entire function of $\lambda \in \mathfrak{a}_\mathbb{C}^*$. In particular, the possible λ-singularities of $\varphi_\emptyset(m;\lambda,h)$ have the same location and not higher order than those of $n_\emptyset^-(m;\lambda)$.

Proof. The denominator function $d^-_\emptyset(m;\lambda)$ contributes to $\varphi_\emptyset(m;\lambda,h)$ simple zeros along the hyperplanes $\mathcal{H}_{n,\alpha}$ with $\alpha \in \Sigma^+_{\mathrm{i}}$ and $n \in \mathbb{N}$. They cancel the possible single poles of the Harish-Chandra series $\Phi(m;\lambda,h)$ predicted by Theorem 1.3.4. □

Our first goal is to generalize Proposition 1.5.3 to arbitrary $\Theta \subset \Pi$ and prove that for $h \in A^+U$ the possible singularities of the Θ-spherical function $\varphi_\Theta(m;\lambda,h)$ are those of the numerator function $n^-_\Theta(m;\lambda)$. We begin with a simple "counting argument" for the possible location and order of the singularities in λ when $m \in \mathcal{M}$ and $h \in A^+U$ are fixed.

Lemma 1.5.4 *Suppose $m \in \mathcal{M}$ and $h \in A^+U$ are fixed. Then possible singularities of $\varphi_\Theta(m;\lambda,h)$ in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ are those of $n^-_\Theta(m;\lambda)$ together with possible simple poles along the hyperplanes*

$$\mathcal{H}_{n,\alpha}, \qquad \text{with } \alpha \in \langle\Theta\rangle^+_{\mathrm{i}} \text{ and } n \in \mathbb{Z}.$$

Proof. Notice first that $wH_\alpha = H_{w\alpha}$ for all $\alpha \in \Sigma$ and $w \in W$. Hence $w\lambda \in \mathcal{H}_{n,\alpha}$ if and only if $\lambda \in \mathcal{H}_{n,w^{-1}\alpha}$.

Let $w \in W_\Theta$. According to Lemma 1.4.9 (a), the singularities of the function $c^+_\Theta(m;w\lambda)$ are at most simple poles located along the hyperplanes $\mathcal{H}_{-n,w^{-1}\alpha}$ with $\alpha \in \langle\Theta\rangle^+_{\mathrm{i}}$ and $n \in \mathbb{N}_0$. By Theorem 1.3.4, for every $h \in A^+U$, the function $\Phi(m;w\lambda,h)$ has at most simple poles along the hyperplanes $\mathcal{H}_{n,w^{-1}\alpha}$ with $\alpha \in \Sigma^+_{\mathrm{i}}$ and $n \in \mathbb{N}$. Moreover, $w(\langle\Theta\rangle) \subset \langle\Theta\rangle$ and $w(\Sigma^+_{\mathrm{i}} \setminus \langle\Theta\rangle^+_{\mathrm{i}}) \subset \Sigma^+_{\mathrm{i}} \setminus \langle\Theta\rangle^+_{\mathrm{i}}$ by Lemma 1.1.2. Therefore, for every fixed $h \in A^+U$, the singularities of $\sum_{w\in W_\Theta} c^+_\Theta(m;w\lambda)\Phi(m;w\lambda,h)$ are at most simple poles located along the hyperplanes

$$\begin{aligned} &\mathcal{H}_{n,\alpha}, && \text{with } \alpha \in \langle\Theta\rangle^+_{\mathrm{i}} \text{ and } n \in \mathbb{Z},\\ &\mathcal{H}_{n,\alpha}, && \text{with } \alpha \in \Sigma^+_{\mathrm{i}} \setminus \langle\Theta\rangle^+_{\mathrm{i}} \text{ and } n \in \mathbb{N}. \end{aligned}$$

This proves the lemma, since the denominator function $d^-_\Theta(m;\lambda)$ contributes to $\varphi_\Theta(m;\lambda,h)$ with simple zeros along the hyperplanes $\mathcal{H}_{n,\alpha}$ with $\alpha \in \Sigma^+_{\mathrm{i}} \setminus \langle\Theta\rangle^+_{\mathrm{i}}$ and $n \in \mathbb{N}$. □

Using the method of pole cancellation developed by Opdam in [Opd88a], we now prove that the singularities along hyperplanes associated with roots in $\langle\Theta\rangle^+_{\mathrm{i}}$ are in fact removable. Since this property is trivially satisfied when $\Theta = \emptyset$, we shall exclude this case from the following discussion. To put ourselves in the setting in which Opdam's method applies, we need the following lemma.

Lemma 1.5.5 *Let $m \in \mathcal{M}$ and $h \in A^+U$. Then the singularities of $\varphi_\Theta(m;\lambda,h)$ along the hyperplanes $\mathcal{H}_{n,\alpha}$ with $\alpha \in \langle\Theta\rangle^+_{\mathrm{i}}$ and $n \in \mathbb{Z}$ are removable if and only if the singularities along the same hyperplanes of the function*

$$\varphi^\#_\Theta(m;\lambda,h) := \sum_{w\in W_\Theta} c^+_\Pi(m;w\lambda)\Phi(m;w\lambda,h)$$

are removable.

Proof. Recall the function $c_{\Theta}^{+,c}$ introduced in Definition 1.4.1. It is W_{Θ}-invariant by Lemma 1.4.4. Hence for all $w \in W_{\Theta}$, we have $c_{\Pi}^{+}(m; w\lambda) = c_{\Theta}^{+,c}(m; \lambda) c_{\Theta}^{+}(m; w\lambda)$ and

$$\begin{aligned} c_{\Theta}^{-}(m; \lambda)\varphi_{\Theta}^{\#}(m; \lambda, h) &= c_{\Theta}^{-}(m; \lambda) \sum_{w \in W_{\Theta}} c_{\Pi}^{+}(m; w\lambda)\Phi(m; w\lambda, h) \\ &= c_{\Theta}^{+,c}(m; \lambda)\varphi_{\Theta}(m; \lambda, h). \end{aligned}$$

The possible zeros and poles of $c_{\Theta}^{+,c}$ and c_{Θ}^{-} are located along hyperplanes associated with roots in $\Sigma_{\mathrm{i}}^{+} \setminus \langle\Theta\rangle_{\mathrm{i}}^{+}$. The intersection of a hyperplane associated with a root in $\langle\Theta\rangle_{\mathrm{i}}^{+}$ and a hyperplane associated with a root in $\Sigma_{\mathrm{i}}^{+} \setminus \langle\Theta\rangle_{\mathrm{i}}^{+}$ is a variety of codimension > 1. The claim therefore follows from Hartogs theorem (see e.g. [Ran86], E.1.3, p. 47). □

Proposition 1.5.6 *Let $\emptyset \neq \Theta \subset \Pi$. Suppose $m \in \mathcal{M}$ and $h \in A^{+}U$. The singularities of $\varphi_{\Theta}(m; \lambda, h)$ along the hyperplanes $\mathcal{H}_{n,\alpha}$ with $\alpha \in \langle\Theta\rangle_{\mathrm{i}}^{+}$ and $n \in \mathbb{Z}$ are all removable.*

Proof. By Lemma 1.5.5, it is enough to prove the removability of these singularities for the function $\varphi_{\Theta}^{\#}(m; \lambda, h)$. We argue as in the proof of Corollary 2.7 in [Opd88a] (see also Theorem 4.3.14 in [HS94]). The function $\varphi_{\Theta}^{\#}(m; \lambda, h)$ cannot have simple poles along hyperplanes of the form $\mathcal{H}_{0,\alpha}$ with $\alpha \in \langle\Theta\rangle^{+}$ because it is W_{Θ}-invariant. By W_{Θ}-invariance, we can also restrict ourselves to prove the removability of the poles along $\mathcal{H}_{n,\alpha}$ with $\alpha \in \langle\Theta\rangle^{+}$ and $n \in \mathbb{N}$.

Let then $\alpha \in \langle\Theta\rangle^{+}$ and $n \in \mathbb{N}$ be fixed. We choose λ_0 in $\mathcal{H}_{n,\alpha}$ with the property that λ_0 does not belong to any other hyperplane $\mathcal{H}_{m,\beta}$ with $\beta \in \langle\Theta\rangle_{\mathrm{i}}^{+}$ and $m \in \mathbb{Z}$. We claim that for $h \in A^{+}U$ the residue

$$\operatorname*{Res}_{\lambda_0} \varphi_{\Theta}^{\#}(m; \lambda, h) := \lim_{\lambda \to \lambda_0} \{\langle \lambda - n\alpha, n\alpha\rangle \, \varphi_{\Theta}^{\#}(m; \lambda, h)\}$$

of $\varphi_{\Theta}^{\#}(m; \lambda, h)$ along $\mathcal{H}_{n,\alpha}$ vanishes at λ_0. Let r_α denote the reflection associated with the root α. As in [Opd88a], Corollary 2.7 (second case), we compute

$$\operatorname*{Res}_{\lambda_0} \varphi_{\Theta}^{\#}(m; \lambda, h) = \sum_{\substack{w \in W_{\Theta} \\ w\alpha > 0}} d(m, w; n, \alpha, \lambda_0) \; \Phi(m; w r_\alpha \lambda_0, h)$$

with coefficients

$$d(m, w; n, \alpha, \lambda_0) = \lim_{\lambda \to \lambda_0} \langle \lambda - n\alpha, n\alpha\rangle \; \{c_{\Pi}^{+}(m; w r_\alpha \lambda) + c_{\Pi}^{+}(m; w\lambda_0) \; \Gamma_{2nw\alpha}(m; w\lambda)\}.$$

Up to a constant multiple, these coefficients agree with those in [Opd88a], and they are hence equal to zero. Notice that, since $\varphi_{\Theta}^{\#}(m; \lambda, h)$ possesses at most simple poles, the singularity along the hyperplane $\mathcal{H}_{n,\alpha}$ is removable provided the residue of $\varphi_{\Theta}^{\#}(m; \lambda, h)$ at each point along this hyperplane vanishes. Hartogs theorem then ensures that it is sufficient to check only the vanishing of the residues at points λ_0 which belong to no other singular hyperplane than $\mathcal{H}_{n,\alpha}$. □

Corollary 1.5.7 *Suppose $m \in \mathcal{M}$ and $h \in A^+U$ are fixed. Then the function*

$$\frac{\varphi_\Theta(m;\lambda,h)}{n_\Theta^-(m;\lambda)}$$

extends as a W_Θ-invariant entire function of $\lambda \in \mathfrak{a}_\mathbb{C}^*$. In particular, the polar set of $\varphi_\Theta(m;\lambda,h)$ is contained (counting multiplicities) in the polar set of the function $n_\Theta^-(m;\lambda)$.

When $\Theta = \Pi$, then $n_\Pi^- \equiv 1$ and the function $\varphi_\Pi(m;\lambda,h)$ is W-invariant and entire in $\lambda \in \mathfrak{a}_\mathbb{C}^*$.

If Σ is reduced, then $\varphi_\Theta(m;\lambda,h)$ has at most simple poles.

Proof. The first part of the corollary is an immediate consequence of Lemma 1.5.4 and Proposition 1.5.6. The last statement follows from Lemma 1.4.9. □

We now turn our attention to the dependence on the variable $h \in A^+U$.

Corollary 1.5.8 *Let $m \in \mathcal{M}$, and let $\lambda \in \mathfrak{a}_\mathbb{C}^*$ be generic. Then $\varphi_\Theta(m;\lambda,h)$ is a holomorphic function of $h \in A^+U$, where U is the open subset of T from Definition 1.5.1.*

Proof. This follows immediately from Theorem 1.3.4. □

We wish to construct out of $\varphi_\Theta(m;\lambda,h)$ a W_Θ-invariant function in the variable h. The extension by W_Θ-invariance to a neighborhood of $W_\Theta(A^+)$ in $A_\mathbb{C}$ is always possible since $W_\Theta(A^+)$ is the disjoint union of the family $\{w(A^+) : w \in W_\Theta\}$. The problem is the extension along the "walls" separating different regions $w(A^+)$ – recall that the walls of each Weyl chamber $w(\mathfrak{a}^+)$ are the portions of hyperplanes $\mathcal{H}_{0,\alpha}$, $\alpha \in \Sigma^+$, forming the boundary of each $w(\mathfrak{a}^+)$. By abuse of language, we shall also speak of *walls* in A –. The required extension turns out to be always possible. Indeed, the summation over W_Θ in the definition of the Θ-spherical functions cancels all singularities on the walls corresponding to elements of Θ. In the easy example of $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ this is obvious: the functions $\Phi(2;\lambda,z) = e^{\lambda z}/2\sinh z$ has no holomorphic extension at the point $z = 0$, but for the weighted sum $\varphi_\Pi(2;\lambda,z) = (e^{\lambda z} - e^{-\lambda z})/(2\lambda\sinh z)$, the singularity at the point $z = 0$ is removable.

To prove similar results in the general case, one needs the monodromy techniques developed by Heckman and Opdam in the case of $\Theta = \Pi$. The key feature of these techniques is a sort of rank-one reduction. One removes the singularities along a given wall of A^+ by forming a weighted sum of the function and its reflection across that wall. When the sum is over the entire Weyl group, all possible walls in A^+ can be considered, and the resulting function turns out to be holomorphic on a tubular neighborhood of A in $A_\mathbb{C}$. When the sum is only on a smaller parabolic subgroup W_Θ, one can only use reflections associated with elements of Θ. The holomorphic extension is therefore only obtained in a tubular neighborhood in $A_\mathbb{C}$ of the largest W_Θ-invariant open subset A_Θ of A containing A^+. The precise argument to remove the singularities along the walls depends on the theory of holonomic systems of differential equations with regular singularities. Let r_j denote the reflection across the wall

of A^+ corresponding to $\alpha_j \in \Pi$. Let $a \in A^+$ and let $z_a = Wa$ its image in $W \setminus A_{\mathbb{C}}^{\mathrm{reg}}$. A local solution f at z_a for the hypergeometric system can be analytically continued along a loop g_j in $W \setminus A_{\mathbb{C}}^{\mathrm{reg}}$ based at z_a. Since the hypergeometric system has holomorphic coefficients in $W \setminus A_{\mathbb{C}}^{\mathrm{reg}}$, the resulting function $\mu(g_j)(f)$ is again a local solution of the hypergeometric system. Moreover, $\mu(g_j)(f)$ depends only on f and on the homotopy class of g_j in the fundamental group $\Pi_1(W \setminus A_{\mathbb{C}}^{\mathrm{reg}}, z_a)$. The direct computation by Heckman and Opdam of the monodromy action of $\mu(g_j)$ on the Harish-Chandra series implies that

$$c_{\Pi}^{+}(m;\lambda)\Phi(m;\lambda) + c_{\Pi}^{+}(m;r_j\lambda)\Phi(m;r_j\lambda) \tag{1.51}$$

is fixed by $\mu(g_j)$. Since (1.51) has moderate growth along the wall corresponding to $\alpha_j \in \Pi$, it follows that its singularity along that wall is removable. We shall apply this argument to each $\alpha_j \in \Theta$.

Before further proceeding towards the main result, we briefly study the properties of the set A_Θ.

Let $\overline{\mathfrak{a}^+} := \{H \in \mathfrak{a} : \alpha(H) \geq 0 \text{ for all } \alpha \in \Sigma^+\}$ denote the closure of the positive Weyl chamber, and let $\overline{A^+} := \exp(\overline{\mathfrak{a}^+})$. We define

$$\mathfrak{a}_\Theta := \left(W_\Theta \cdot \overline{\mathfrak{a}^+}\right)^0 \quad \text{and} \quad A_\Theta := \left(W_\Theta \cdot \overline{A^+}\right)^0, \tag{1.52}$$

where ^{0} denotes the interior.

Example 1.5.9 Let Σ be of type A_2 with $\Pi = \{\alpha_1, \alpha_2\}$ and $\Sigma^+ = \{\alpha_1, \alpha_2, \widetilde{\alpha} := \alpha_1 + \alpha_2\}$. Then $\mathfrak{a}_\emptyset = \mathfrak{a}^+$ and $\mathfrak{a}_\Pi = \mathfrak{a}$. The domain $\mathfrak{a}_\Theta$ in Figure 1.1 corresponds to $\Theta = \{\alpha_1\}$.

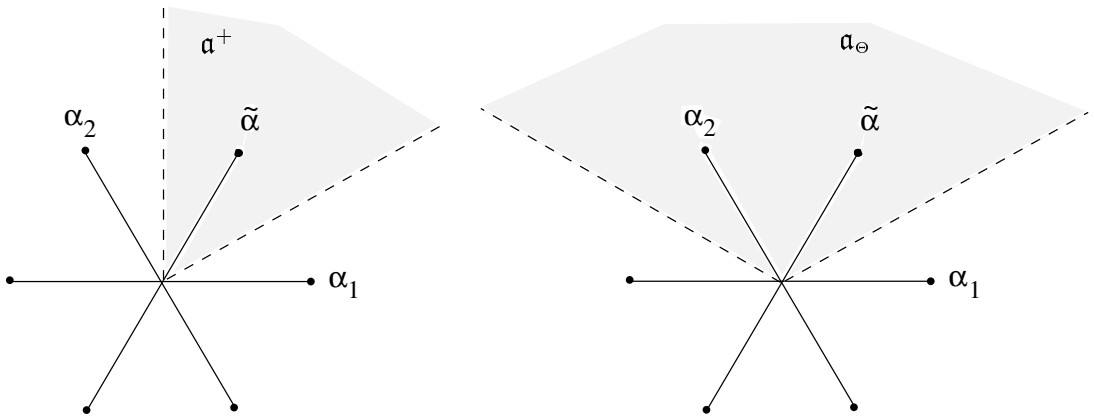


Figure 1.1: Example of $\mathfrak{a}_\Theta$ for the system of type A_2

The set

$$\overline{\mathfrak{a}_\Theta} = W_\Theta(\overline{\mathfrak{a}^+}) = \cup_{w \in W_\Theta} w(\overline{\mathfrak{a}^+}) \tag{1.53}$$

is the smallest cone (i.e. $\mathbb{R}^+$-invariant set) in $\mathfrak{a}$ which is closed and W_Θ-invariant and contains $\overline{\mathfrak{a}^+}$. It is a union of closed Weyl chambers. The open polyhedral cone in $\mathfrak{a}^*$

$$C_\Theta := \sum_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \mathbb{R}^+ H_\alpha$$

has dual cone

$$\begin{aligned} C_\Theta^* :&= \{H \in \mathfrak{a} : \langle H, Y\rangle \geq 0 \text{ for all } Y \in C_\Theta\} \\ &= \{H \in \mathfrak{a} : \alpha(H) \geq 0 \text{ for all } \alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+\}. \end{aligned}$$

As intersection of the closed hyperplanes defined by roots, C_Θ^* is a convex cone which is a union of closed Weyl chambers in $\mathfrak{a}$.

Lemma 1.5.10 *We have*

$$\overline{\mathfrak{a}_\Theta} = C_\Theta^*.$$

Consequently, $\overline{\mathfrak{a}_\Theta}$ is a closed W_Θ-invariant convex cone in $\mathfrak{a}$ and also its interior $\mathfrak{a}_\Theta$ is convex.

Proof. The cone C_Θ^* is W_Θ-invariant. Indeed for all $H \in C_\Theta^*$, $\alpha_j \in \Theta$ and $\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$

$$\alpha(r_j(H)) = (r_j\alpha)(H) \geq 0$$

because $r_j\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$ (see Lemma 1.1.2). Hence $r_j(H) \in C_\Theta^*$. Moreover, $\overline{\mathfrak{a}^+} \subset C_\Theta^*$. Thus $\overline{\mathfrak{a}_\Theta} \subset C_\Theta^*$.

Suppose now $\overline{\mathfrak{a}_\Theta} \neq C_\Theta^*$. Observe that both cones are unions of closed Weyl chambers. Moreover C_Θ^* is convex. Hence there is a closed Weyl chamber $C \subset C_\Theta^* \setminus \mathfrak{a}_\Theta$ such that $\overline{C} \cap \overline{\mathfrak{a}_\Theta}$ is the wall F of a chamber in $\overline{\mathfrak{a}_\Theta}$. Let α be the element in Σ^+ such that $\alpha(H) = 0$ for all $H \in F$, and let C' be the Weyl chamber of $\overline{\mathfrak{a}_\Theta}$ with wall F. If $\alpha \in \langle\Theta\rangle^+$, then $C = r_\alpha(C') \subset W_\Theta(\overline{\mathfrak{a}^+}) = \overline{\mathfrak{a}_\Theta}$. Thus $\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$. Let now $H \in C^0$. Then $r_\alpha(H) \in W_\Theta(\mathfrak{a}^+)$. Hence there is $H' \in \mathfrak{a}^+$ and $w_0 \in W_\Theta$ such that $r_\alpha(H) = w_0(H')$. Since $w_0^{-1}\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$, we have

$$\alpha(r_\alpha(H)) = \alpha(w_0(H')) = (w_0^{-1}\alpha)(H') > 0.$$

H and $r_\alpha(H)$ lie on opposite sides of the hyperplane $\{H \in \mathfrak{a} : \alpha(H) = 0\}$. Therefore $\alpha(H) < 0$ and $H \notin C_\Theta^*$, against the choice of C. This contradiction thus implies that $\overline{\mathfrak{a}_\Theta} = C_\Theta^*$.

The cone $\mathfrak{a}_\Theta$ is convex because the interiors of convex sets are convex (cf. e.g. [Roc70], Theorem 6.1, p. 45). □

The main result of this section on the regularity properties of the Θ-spherical functions is the following theorem.

Theorem 1.5.11 *For a fixed multiplicity function m, there exists a W_Θ- invariant tubular neighborhood U_Θ of A_Θ in $A_{\mathbb{C}}$ on which the function*

$$\frac{\varphi_\Theta(m;\lambda,h)}{n_\Theta^-(m;\lambda)} = \frac{1}{d_\Theta^-(m;\lambda)} \sum_{w\in W_\Theta} c_\Theta^+(m;w\lambda)\Phi(m;w\lambda;h)$$

extends as a W_Θ-invariant holomorphic function of $(\lambda, h) \in \mathfrak{a}_{\mathbb{C}}^* \times U_\Theta$. In particular, the λ-singularities of the Θ-spherical function $\varphi_\Theta(m;\lambda,h)$ are contained (counted with multiplicities) in the polar set of the numerator function $n_\Theta^-(m;\lambda)$. The λ-singularities are simple poles when Σ is a reduced root system.

The function $\varphi_\Theta(m;\lambda,h)/n_\Theta^-(m;\lambda)$ is entire in $m \in \mathcal{M}$.

Proof. The argument follows the lines of Corollary 4.3.9 in [HS94]. Let r_j denote the reflection associated with the simple root $\alpha_j \in \Theta$. If $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is regular (i.e. if $\lambda_\alpha \notin \mathbb{Z}$ for all $\alpha \in \Sigma^+$), then Theorem 4.3.6 in [HS94] ensures that there is a neighborhood of $\overline{A^+ \cup r_j A^+}^{\,0}$ in $A_{\mathbb{C}}$ on which

$$c_{\Pi}^+(m;w\lambda)\Phi(m;w\lambda,h) + c_{\Pi}^+(m;r_jw\lambda)\Phi(m;r_jw\lambda,h)$$

extends as a holomorphic r_j-invariant function. Set

$$W_\Theta(j) := \{w \in W_\Theta : w^{-1}\alpha_j \in \langle\Theta\rangle^+\}.$$

Then W_Θ is the disjoint union

$$W_\Theta = W_\Theta(j) \coprod r_j\big(W_\Theta(j)\big).$$

Indeed, if $w \in W_\Theta \setminus W_\Theta(j)$, then $(r_jw)^{-1}\alpha_j = -w^{-1}\alpha_j \in \langle\Theta\rangle^+$. Hence, for all regular $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ which are not zeros of $c_\Theta^{c,+}(m;\lambda)$,

$$\sum_{w\in W_\Theta} c_\Theta^+(m;w\lambda)\Phi(m;w\lambda,a) =$$

$$= \frac{1}{c_\Theta^{c,+}(m;\lambda)} \sum_{w\in W_\Theta(j)} \big[\, c_{\Pi}^+(m;w\lambda)\Phi(m;w\lambda,h) + c_{\Pi}^+(m;r_jw\lambda)\Phi(m;r_jw\lambda;h)\,\big]$$

extends to a holomorphic r_j-invariant function on some neighborhood of $\overline{A^+ \cup r_j A^+}^{\,0}$ in $A_{\mathbb{C}}$. Since α_j is arbitrary in Θ, we conclude that the function $\varphi_\Theta(m;\lambda,h)/n_\Theta^-(m;\lambda)$ is a holomorphic in h on a tubular neighborhood U_Θ of A_Θ in $A_{\mathbb{C}}$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ which do not lie on a locally finite union of hyperplanes. The result then follows from Corollary 1.5.7, the W_Θ-invariance in (λ, h), and Lemma 2.6 in [Opd88a]. Finally, recall from Theorem 1.3.4 that $\Phi(m;\lambda,h)$ is an entire function of m by (1.37) and (1.46). Moreover, $c_\Theta^+(m;\lambda)$ and$1/d_\Theta^-(m;\lambda)$ are entire in m. Thus so is also $\varphi_\Theta(m;\lambda,h)/n_\Theta^-(m;\lambda)$. □

Example 1.5.12 Since the root system of a NCC symmetric space G/H is always reduced, the singularities in the spectral parameter λ of the spherical functions on G/H are always simple poles. They are contained in the polar set of the numerator function

$$n_\Omega(\lambda) = \prod_{\alpha\in\Sigma_+} \Gamma(-\lambda_\alpha - m_\alpha/2 + 1).$$

Corollary 1.5.13 *Retain the notation of Theorem 1.5.11 and let $(m,h) \in \mathcal{M} \times U_\Theta$ be fixed. Then the function*

$$\frac{\varphi_\Theta(m;\lambda,h)}{c_\Theta^-(m;\lambda)}$$

is holomorphic in λ on the open set

$$B_\Theta := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \Re\lambda_\alpha < 1 \quad \textit{for all } \alpha \in \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+\}.$$

Proof. We have

$$\frac{\varphi_\Theta(m;\lambda,h)}{c_\Theta^-(m;\lambda)} = \frac{\varphi_\Theta(m;\lambda,h)}{n_\Theta^-(m;\lambda)}\, d_\Theta^-(m;\lambda)$$

with $\varphi_\Theta(m;\lambda,h)/n_\Theta^-(m;\lambda)$ entire in λ by Theorem 1.5.11. Moreover, the singularities of $d_\Theta^-(m;\lambda)$ are located along the hyperplanes $\mathcal{H}_{n,\alpha} = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \lambda_\alpha = n\}$ for $\alpha \in \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+$ and $n \in \mathbb{N}$. □

Example 1.5.14 (The rank-one case) In the rank-one case, there are only two possibilities for Θ, namely $\Theta = \Pi$ or $\Theta = \emptyset$. The first case yields constant multiples of the *Jacobi functions of first type*

$${}_2F_1\left(\frac{m_\alpha + 2m_{2\alpha} + 2\lambda}{4}, \frac{m_\alpha + 2m_{2\alpha} - 2\lambda}{4}; \frac{m_\alpha + 2m_{2\alpha} + 1}{2}; -\sinh^2 t\right).$$

In the case $\Theta = \emptyset$, the function $\varphi_\emptyset$ is a multiple of the Harish-Chandra series. We have already remarked on p. 34 that the possible λ-singularities of $\varphi_\emptyset$ are in this case the poles of (1.50). The study of the singular set of the hypergeometric function is very classical (see e.g. [vV02]). For the geometric case of the NCC symmetric space $\mathrm{SO}_0(1,n)/\mathrm{SO}_0(1,n-1)$ with $n \geq 2$, the exact list of singularities of $\varphi_\emptyset$ has been quickly determined with the method of Bernstein-Sato polynomials in [ÓP01a], Corollary 6.4. From the obtained list, one sees that the λ-singularities of $\varphi_\emptyset$ are exactly all (simple) poles of the functions $n_\emptyset^-$ when m_α is odd. When m_α is even, there are only finitely many poles (and they are all simple). As we shall see in the following section, the fact that there are only finitely many singular hyperplanes is a feature of the Θ-spherical functions for all triples $(\mathfrak{a},\Sigma,m)$ with Σ reduced and m_α even for all $\alpha \in \Sigma$.

1.6 Even multiplicities

In this section we investigate the regularity properties of Θ-spherical functions for triples $(\mathfrak{a},\Sigma,m)$ in which Σ is a reduced root system and m is an even positive multiplicity function.

Definition 1.6.1 Let Σ be a reduced root system. A multiplicity function m on Σ is said to be *even* if $m_\alpha \in 2\mathbb{Z}$ for all $\alpha \in \Sigma$. An even multiplicity function is *positive* if $m_\alpha \in 2\mathbb{N}$ for all $\alpha \in \Sigma$. We denote by $\mathcal{E}$ the set of even multiplicity functions on Σ and by $\mathcal{E}^+$ the subset of even positive multiplicity functions.

Observe that $2\rho(m) \in P$ for all $m \in \mathcal{E}$, where P denotes the restricted weight lattice from Definition 1.1.4.

Let $\Sigma = \coprod_{j=1}^{q} \Omega_j$ be the disjoint union of the W-orbits in Σ. For $j = 1, \ldots, q$ we define $m_j \in \mathcal{E}$ by

$$m_j(\alpha) := \begin{cases} 2 & \text{if } \alpha \in \Omega_j \\ 0 & \text{otherwise.} \end{cases} \tag{1.54}$$

In words, m_j is twice the characteristic function of the orbit Ω_j. Recall that multiplicity functions are W-invariant by definition. Hence every even multiplicity function can be written as a linear combination of the elements $\{m_j : j = 1, \ldots, q\}$ with integral coefficient. The positive even multiplicities have coefficients in $\mathbb{N}$. In particular, the set $\mathcal{E}$ (respectively, $\mathcal{E}^+$) is the $\mathbb{Z}$-lattice (respectively, $\mathbb{N}$-semi-lattice) generated by $\{m_j : j = 1, \ldots, q\}$.

Example 1.6.2 (The Riemannian case) In the geometric case, even multiplicities correspond to Riemannian spaces G/K with the property that all Cartan subalgebras in the Lie algebra $\mathfrak{g}$ of G are conjugate under the adjoint group of $\mathfrak{g}$ (see [Hel78], p. 429). The easiest examples correspond to the complex cases, in which all multiplicities are equal to 2. According to classification, the irreducible Riemannian symmetric spaces with even multiplicities are $\mathrm{SO}_0(2n+1,1)/\mathrm{SO}(2n+1)$, $\mathrm{SU}^*(2n)/\mathrm{Sp}(n)$, $E_{6(-26)}/F_{4(-52)}$ and the irreducible spaces of the form $G_{\mathbb{C}}/U$, where U is a compact real form of $G_{\mathbb{C}}$. The latter spaces admit a complex structure.

Example 1.6.3 (The NCC case) Up to coverings, the irreducible NCC spaces with even multiplicities are $\mathrm{SO}_0(2n+1,1)/\mathrm{SO}_0(2n,1)$, $\mathrm{SU}^*(2(p+q))/\mathrm{Sp}(p,q)$, $E_{6(-26)}/F_{4(-20)}$ and the irreducible spaces $G_{\mathbb{C}}/G$, where G is Hermitian. The spaces $G_{\mathbb{C}}/G$ with G Hermitian are all endowed with a complex structure. We refer to [AÓS00] for more information on these spaces and their harmonic analysis.

The key tool for studying Θ-spherical functions for positive even multiplicity functions are Opdam's shift operators [Opd88a]. See also [HS94], Section 3.1. The adjective "shift" emphasizes their property of translating the multiplicity parameter by an element of $\mathcal{E}$. Translations by non-zero elements of $\mathcal{E}$ with $m_\alpha \geq 0$ for all $\alpha \in \Sigma$ correspond to the so-called *raising operators*, whereas translations by non-zero elements of $\mathcal{E}$ with $m_\alpha \leq 0$ for all $\alpha \in \Sigma$ correspond to the *lowering operators*.

A shift operator $D(l)$ corresponding to the translation parameter $l \in \mathcal{E}$ maps the Harish-Chandra series $\Phi(m;\lambda,h)$ into the multiple $\eta(m;\lambda)\Phi(m+l;\lambda,h)$ of the Harish-Chandra series $\Phi(m+l;\lambda,h)$ corresponding to the shifted multiplicity function $m+l$. Here $\eta(m;\lambda)$ denotes a rational function of m and λ. In this way we can obtain the Harish-Chandra series $\Phi(m;\lambda,h)$ for an even multiplicity case as image under a suitable shift operator of the Harish-Chandra series $\Phi(0;\lambda,h) = e^{\lambda(\log h)}$ corresponding to the Euclidean case $m = 0$. The analytic properties of $\Phi(m;\lambda,h)$ reflect therefore those of the shift operator.

Definition 1.6.4 Let $\mathbb{C}[\mathcal{M}]\otimes\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})$ denote the algebra of differential operators on $A_{\mathbb{C}}$ with coefficients in $\mathbb{C}[A_{\mathbb{C}}^{\mathrm{reg}}]$ depending polynomially on the multiplicities. Recall from (1.13) the notation $D(p_L,m) := L(m) + \langle\rho(m),\rho(m)\rangle$. A *shift operator with even shift* $l \in \mathcal{E}$ is a differential operator $D(l) \in \mathbb{C}[\mathcal{M}] \otimes \mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})$ satisfying the following properties:

(i) For all $m \in \mathcal{M}$

$$D(l)\circ D(p_L,m) = D(p_L,m+l)\circ D(l);$$

(ii) the differential operator $D(l)$ admits on A^+ an asymptotic expansion of the form

$$D(l) = \sum_{\mu\in 2\Lambda} e^{-\rho(l)-\mu}\partial(p_\mu)$$

with $p_\mu \in \mathbb{C}[\mathcal{M}\times\mathfrak{a}_{\mathbb{C}}]$.

We refer to Chapter 3 in [HS94] for the algebraic properties of the shift operators. See also Section 5 in [Opd00].

Lemma 1.6.5 *Let m be a fixed even multiplicity function. Let U be the open subset of T from Theorem 1.3.4. Then there exists a shift operator $D(m)$ so that*

$$D(m)e^{\lambda(\log h)} = [c_{\Pi}^{+}(m;-\lambda)]^{-1}\Phi(m;\lambda,h) \tag{1.55}$$

for all generic $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and all $h \in A^+U$.

Proof. Since $\Phi(0;\lambda,h) = e^{\lambda(\log h)}$ and $c_{\Pi}^{+}(0,-\lambda) = 1$, the shift operator $D(m)$ is the raising operator constructed in Corollary 3.4.4 in [HS94]. □

Let V be a tubular neighborhood of A in $A_{\mathbb{C}}$ for which the function $(\lambda,h) \mapsto e^{\lambda(\log h)}$ is single valued and holomorphic on $\mathfrak{a}_{\mathbb{C}}^*\times V$. Setting $V^{\mathrm{reg}} := V\cap A_{\mathbb{C}}^{\mathrm{reg}}$, we conclude that $(\lambda,h)\mapsto D(m)e^{\lambda(\log h)}$ is holomorphic in $\mathfrak{a}_{\mathbb{C}}^*\times V^{\mathrm{reg}}$.

Equality (1.55) will allow us to sharpen the result in Theorem 1.5.11 by showing that the λ-singularities of Θ-spherical functions corresponding to a reduced root system with an even multiplicity function are located on a specific *finite* union of affine hyperplanes.

Let us first make a simple observation. For every $\alpha\in\Sigma$, consider the functions $c_\alpha^+(m;\lambda)$ and $c_\alpha^-(m;\lambda)$ from Definition 1.4.1. Since Σ is reduced and m is even, part (d) of Remark 1.4.2 gives

$$c_\alpha^+(m;-\lambda) = \prod_{k=0}^{m_\alpha/2-1}\frac{1}{-\lambda_\alpha+k} = (-1)^{m_\alpha/2}\prod_{k=0}^{m_\alpha/2-1}\frac{1}{\lambda_\alpha-k},$$

$$c_\alpha^-(m;\lambda) = \prod_{k=-m_\alpha/2+1}^{0}\frac{1}{-\lambda_\alpha+k} = (-1)^{m_\alpha/2}\prod_{k=-m_\alpha/2+1}^{0}\frac{1}{\lambda_\alpha-k}.$$

Therefore

$$c_\alpha^-(m;\lambda)\,c_\alpha^+(m;-\lambda) = \left[\lambda_\alpha\prod_{k=-m_\alpha/2+1}^{m_\alpha/2-1}(\lambda_\alpha-k)\right]^{-1}. \tag{1.56}$$

Theorem 1.6.6 *Let m be an even multiplicity function on a reduced root system Σ. Define for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $\Theta \subset \Pi$*

$$e_\Theta^-(m;\lambda) := \prod_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \prod_{k=-m_\alpha/2+1}^{m_\alpha/2-1} (\lambda_\alpha - k). \tag{1.57}$$

Then there is a W_Θ-invariant tubular neighborhood U_Θ in $A_{\mathbb{C}}$ of the set A_Θ of (1.52) such that the function

$$e_\Theta^-(m;\lambda)\ \varphi_\Theta(m;\lambda,h)$$

extends as a W_Θ-invariant holomorphic function of $(\lambda,h) \in \mathfrak{a}_{\mathbb{C}}^* \times U_\Theta$.

Proof. Observe that $c_\Pi^+ = c_\Theta^+ c_\Theta^{+,c}$. By Lemma 1.4.4, the function $c_\Theta^{+,c}$ is W_Θ-invariant. Substitution of (1.55) and (1.56) in (1.48) gives for all generic $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and for $h \in A^+U$

$$\begin{aligned}
\varphi_\Theta(m;\lambda,a) &= c_\Theta^-(m;\lambda) \sum_{w\in W_\Theta} c_\Theta^+(m;w\lambda)c_\Pi^+(m;-w\lambda)D(m)e^{w\lambda(\log h)} \\
&= \Big[\prod_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} c_\alpha^-(m;\lambda)c_\alpha^+(m;-\lambda) \Big] \\
&\quad \times \sum_{w\in W_\Theta} \Big[\prod_{\alpha\in\langle\Theta\rangle^+} c_\alpha^+(m;w\lambda)c_\alpha^+(m;-w\lambda) \Big] D(m)e^{w\lambda(\log h)} \\
&= \prod_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \left[\lambda_\alpha \prod_{k=-m_\alpha/2+1}^{m_\alpha/2-1} (\lambda_\alpha - k) \right]^{-1} \\
&\quad \times \sum_{w\in W_\Theta} \prod_{\alpha\in\langle\Theta\rangle^+} \left[(w\lambda)_\alpha \prod_{k=-m_\alpha/2+1}^{m_\alpha/2-1} \big((w\lambda)_\alpha - k\big) \right]^{-1} D(m)e^{w\lambda(\log a)}.
\end{aligned} \tag{1.58}$$

Recall the notation $\mathcal{H}_{n,\alpha}$ from (1.27). The right-hand side of (1.58) is holomorphic in $h \in V^{\mathrm{reg}}$ and meromorphic in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with possible singularities located along

$$\left[\bigcup_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \bigcup_{k=-m_\alpha/2+1}^{m_\alpha/2-1} \mathcal{H}_{k,\alpha} \right] \cup \left[\bigcup_{w\in W_\Theta} \bigcup_{\alpha\in\langle\Theta\rangle^+} \bigcup_{k=-m_\alpha/2+1}^{m_\alpha/2-1} \mathcal{H}_{k,w^{-1}\alpha} \right].$$

According to Theorem 1.5.11, there is a W_Θ-invariant tubular neighborhood V_Θ of A_Θ such that the left-hand side of (1.58) is holomorphic in $h \in V_\Theta$ and meromorphic in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, with at most simple poles located along

$$\bigcup_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \bigcup_{k=-m_\alpha/2+1}^{\infty} \mathcal{H}_{k,\alpha}.$$

We conclude that for every fixed $h \in V_\Theta \cap V^{\mathrm{reg}}$ the Θ-spherical function $\varphi_\Theta(m;\lambda,h)$ has at most simple poles located along

$$\bigcup_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \bigcup_{k=-m_\alpha/2+1}^{m_\alpha/2-1} \mathcal{H}_{k,\alpha}.$$

The result extends to all $h \in U_\Theta := V_\Theta \cap V$ by Lemma 2.2.11 in [Hör66] (see also Lemma 2.6 in [Opd88b]). □

As an immediate corollary, we deduce the regularity properties of spherical functions on NCC spaces with even multiplicities.

Corollary 1.6.7 *Let φ_λ ($\lambda \in \mathfrak{a}_\mathbb{C}^*$) denote the meromorphically continued spherical functions on a NCC symmetric space G/H with even multiplicities. Define for $\lambda \in \mathfrak{a}_\mathbb{C}^*$*

$$e_\Omega(m;\lambda) := \prod_{\alpha\in\Sigma_+} \prod_{k=-m_\alpha/2+1}^{m_\alpha/2-1} (\lambda_\alpha - k). \tag{1.59}$$

Then there is a W_0-invariant tubular neighborhood U_0 of $S^0 \cap A$ in the complexification of A so that the function $e_\Omega(m;\lambda)\ \varphi_\lambda(a)$ extends as a W_0-invariant holomorphic function of $(\lambda, a) \in \mathfrak{a}_\mathbb{C}^* \times U_0$.

As already remarked, symmetric spaces G/K for which all multiplicities are equal to 2 are precisely those with G admitting a complex structure. The spherical harmonic analysis on Riemannian and NCC spaces with complex G is particularly simplified by explicit formulas for the spherical functions. We now extend these formulas to Θ-spherical functions corresponding to complex cases. Our argument is an easy modification of the procedure used in the Riemannian case (cf. [Hel84], p. 432). The resulting formulas will also provide an independent proof of Theorem 1.6.6 in the complex case.

Let $\varphi_\Theta(2;\lambda,a)$ denote the Θ-spherical function for a triple $(\mathfrak{a},\Sigma,2)$ corresponding to a complex case. By Example 1.3.11, we have $\Phi(2;\lambda,a) = \Delta(a)^{-1}e^{\lambda(\log a)}$, where

$$\Delta = \prod_{\alpha\in\Sigma^+} (e^\alpha - e^{-\alpha})$$

is the Weyl denominator (1.19) for the reduced system Σ. Hence

$$\Delta(a)\varphi_\Theta(2;\lambda,a) = c_\Theta^-(2;\lambda) \sum_{w\in W_\Theta} c_\Theta^+(2;w\lambda)\, e^{w\lambda(\log a)}, \qquad a \in A^+. \tag{1.60}$$

Equation (1.60) extends by analyticity to a tubular neighborhood U of A in $A_\mathbb{C}$.

Lemma 1.6.8 *For all $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $w \in W_\Theta$*

$$c_\Theta^+(2;w\lambda) = \det(w) c_\Theta^+(2;\lambda).$$

Proof. The group W_Θ consists of orthogonal transformations of A with the Euclidean structure induced by $\langle\cdot,\cdot\rangle$. In particular, $\det(w) = \det(w^{-1})$ for all $w \in W_\Theta$. Replacing in (1.60) the variable $a \in A$ with $w^{-1}a$ for $w \in W_\Theta$, we obtain

$$\Delta(w^{-1}a)\varphi_\Theta(2;\lambda,w^{-1}a) = c_\Theta^-(2;\lambda) \sum_{w'\in W_\Theta} c_\Theta^+(2;w'\lambda)\, e^{w'\lambda(\log w^{-1}a)}, \qquad a \in A. \tag{1.61}$$

On the other hand, φ_Θ is W_Θ-invariant in a and the function Δ is skew, that is

$$\Delta(wa) = \det(w)\Delta(a)$$

for all $a \in A$ and $w \in W$ (cf. e.g. [Hel84], p. 193). Hence

$$\begin{aligned}\Delta(w^{-1}a)\varphi_\Theta(2;\lambda,w^{-1}a) &= \det(w)\Delta(a)\varphi_\Theta(2;\lambda,a)\\ &= \det(w)c_\Theta^-(2;\lambda)\sum_{w'\in W_\Theta} c_\Theta^+(2;w'\lambda)\,e^{w'\lambda(\log a)}, \qquad a\in A. \quad (1.62)\end{aligned}$$

Comparison of (1.61) and (1.62) yields on A

$$\begin{aligned}\sum_{w'\in W_\Theta}\det(w)c_\Theta^+(2;w'\lambda)\,e^{w'\lambda} &= \sum_{w'\in W_\Theta} c_\Theta^+(2;w'\lambda)\,e^{w^{-1}w'\lambda}\\ &= \sum_{w'\in W_\Theta} c_\Theta^+(2;ww'\lambda)\,e^{w'\lambda}.\end{aligned}$$

Since the set $\{e^{w\lambda} : w \in W_\Theta\}$ consists of linearly independent functions, we conclude

$$\det(w)c_\Theta^+(2;w'\lambda) = c_\Theta^+(2;ww'\lambda)$$

for all $w' \in W_\Theta$. The lemma is then proven by taking $w' = \mathrm{id} \in W_\Theta$. □

Because of Lemma 1.6.8, we can write (1.60) for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and all h in a tubular neighborhood U_Θ of A in $A_{\mathbb{C}}$ as

$$\Delta(h)\varphi_\Theta(2;\lambda,h) = c_\Theta^-(2;\lambda)c_\Theta^+(2;\lambda)\sum_{w\in W_\Theta}\det(w)\,e^{w\lambda(\log h)}, \qquad (1.63)$$

where

$$c_\Theta^+(2;\lambda) = \Big[\prod_{\alpha\in\langle\Theta\rangle^+}\lambda_\alpha\Big]^{-1} \qquad \text{and} \qquad c_\Theta^-(2;\lambda) = \Big[\prod_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+}(-\lambda_\alpha)\Big]^{-1}$$

are given by (1.41) and (1.42). Consider the factorization $\Delta = \Delta_\Theta^+\Delta_\Theta^-$ with

$$\begin{aligned}\Delta_\Theta^+ &:= \prod_{\alpha\in\langle\Theta\rangle^+}\big(e^\alpha - e^{-\alpha}\big)\\ \Delta_\Theta^- &:= \prod_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+}\big(e^\alpha - e^{-\alpha}\big).\end{aligned}$$

Theorem 1.6.9 *In the complex case, the Θ-spherical functions can be written for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and h in a W_Θ-invariant tubular neighborhood U_Θ of A_Θ in $A_{\mathbb{C}}$ as*

$$\varphi_\Theta(2;\lambda,h) = \frac{c_\Theta^-(2;\lambda)}{\Delta_\Theta^-(h)}\,\varphi_\Theta^0(2;\lambda,h), \qquad (1.64)$$

where

$$\varphi_\Theta^0(2;\lambda,h) := \frac{c_\Theta^+(2;\lambda)}{\Delta_\Theta^+(h)}\sum_{w\in W_\Theta}\det(w)\,e^{w\lambda(\log h)} \qquad (1.65)$$

is the Θ-spherical function with $m_\alpha = 2$ for all $\alpha \in \langle\Theta\rangle$ and $m_\alpha = 0$ for $\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$. The functions

$$\varphi^0_\Theta(2;\lambda,h) \qquad \textit{and} \qquad \Big(\prod_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+}\lambda_\alpha\Big)\Delta^-_\Theta(h)\,\varphi_\Theta(2;\lambda,a)$$

extend as W_Θ-invariant holomorphic functions of $(\lambda,a)\in\mathfrak{a}^*_{\mathbb{C}}\times U_\Theta$.

Proof. In view of (1.63)–(1.64) it suffices to prove the statement on the holomorphy of $\varphi^0_\Theta(2;\lambda,h)$. Because of (1.65), the possible λ-singularities of the function $\varphi^0_\Theta(2;\lambda,h)$ are at most simple poles located along the hyperplanes $\mathcal{H}_{0,\alpha}$ with $\alpha\in\langle\Theta\rangle^+$. These singularities are in fact removable by the W_Θ-invariance in λ. Similarly, the W_Θ-invariance in h proves that the possible simple poles located on the hyperplanes $\{h\in A_{\mathbb{C}} : \alpha(\log h)=0\}$ for $\alpha\in\langle\Theta\rangle^+$ are also removable. □

In the case of a Riemannian symmetric space G/K with G complex (i.e. $\Theta=\Pi$), formula (1.64) reduces to the classical formula (23) in [Hel84], p. 432. For a NCC symmetric space G/H with G complex, (1.64) gives also the relation between spherical functions on G/H and spherical functions on the Riemannian symmetric space $G^{\theta\sigma}/(K\cap H)$.

Chapter 2

The Θ-spherical Fourier transform

2.1 Definition and first properties

Let $(\mathfrak{a}, \Sigma, m)$ be a triple consisting of a Euclidean space $\mathfrak{a}$, a root system $\Sigma \subset \mathfrak{a}^*$ and a multiplicity function m on Σ. Let Σ^+ denote a fixed positive system for Σ with corresponding fundamental system of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_l\}$. We refer to the previous chapter for the unexplained notation.

We shall assume throughout this chapter that $m \in \mathcal{M}^+$. Under this assumption, the function

$$\mu(m) := \prod_{\alpha \in \Sigma^+} \left| e^{\alpha} - e^{-\alpha} \right|^{m_\alpha} \tag{2.1}$$

is continuous on A. On A^+, it agrees with the function δ of (1.22). According to our conventions, we shall write $\mu(m; a)$ for the value of $\mu(m)$ at $a \in A$. Observe that

$$\mu(m) = \prod_{\alpha \in \Sigma} \left| e^{\alpha} - e^{-\alpha} \right|^{m_\alpha/2},$$

hence $\mu(m; a)$ is a W-invariant continuous function of $a \in A$.

Definition 2.1.1 Let $\Theta \subset \Pi$ and let f be a W_Θ-invariant function on $A_\Theta = W_\Theta(\overline{A^+})^0$. The *$\Theta$-spherical Fourier transform of f associated with the data* $(\mathfrak{a}, \Sigma, m)$ is the W_Θ-invariant function $\mathcal{F}_\Theta f(m)$ on $\mathfrak{a}_\mathbb{C}^*$ defined for $\lambda \in \mathfrak{a}_\mathbb{C}^*$ by

$$\begin{aligned} \mathcal{F}_\Theta f(m; \lambda) &:= \frac{1}{|W_\Theta|} \int_{A_\Theta} f(a)\, \varphi_\Theta(m; \lambda, a) \mu(m; a)\, da \\ &= \int_{A^+} f(a) \varphi_\Theta(m; \lambda, a) \delta(m; a)\, da, \end{aligned} \tag{2.2}$$

provided the integral converges. Here da is a suitable normalization of the Haar measure on A.

Let $C_c(A_\Theta)^{W_\Theta}$ denote the set of W_Θ-invariant continuous functions on A_Θ with compact support. Then the Θ-spherical Fourier transform $\mathcal{F}_\Theta f(m)$ is well-defined for all $f \in$

$C_c(A_\Theta)^{W_\Theta}$ because $\varphi_\Theta(m;\lambda,a)$ is a continuous function of $a \in A_\Theta$. In particular, the Θ-spherical Fourier transform is defined on the subspace of C^∞ elements in $C_c(A_\Theta)^{W_\Theta}$, which we indicate by $C_c^\infty(A_\Theta)^{W_\Theta}$.

By Theorem 1.5.11, the *normalized Θ-spherical functions*

$$\varphi_\Theta^-(m;\lambda,a) := \frac{\varphi_\Theta(m;\lambda,a)}{n_\Theta^-(m;\lambda)} \tag{2.3}$$

is W_Θ-invariant and holomorphic in $(\lambda,a) \in \mathfrak{a}_\mathbb{C}^* \times A_\Theta$. This suggests the following normalization for the Θ-spherical transform.

Definition 2.1.2 Let $\Theta \subset \Pi$ and let f be a W_Θ-invariant function on $A_\Theta = W_\Theta(\overline{A^+})^0$. The *normalized Θ-spherical Fourier transform* of f associated with the data $(\mathfrak{a},\Sigma,m)$ is the W_Θ-invariant function $\mathcal{F}_\Theta^- f(m)$ on $\mathfrak{a}_\mathbb{C}^*$ defined for $\lambda \in \mathfrak{a}_\mathbb{C}^*$ by

$$\begin{aligned}\mathcal{F}_\Theta^- f(m;\lambda) &:= \frac{1}{|W_\Theta|}\int_{A_\Theta} f(a)\,\varphi_\Theta^-(m;\lambda,a)\mu(m;a)\,da \\ &= \int_{A^+} f(a)\varphi_\Theta^-(m;\lambda,a)\delta(m;a)\,da,\end{aligned} \tag{2.4}$$

provided the integral converges.

Example 2.1.3

(a) When $\Theta = \Pi$, then $\varphi_\Pi(m;\lambda,a)$ coincides (up to a multiple depending only on the multiplicity function) with Heckman-Opdam's hypergeometric function. Hence $\mathcal{F}_\Theta(m)f$ agrees with the transform of [Opd95] and [HS94], p. 205. In particular, for geometric multiplicities, we recover Harish-Chandra's spherical transform for K-bi-invariant functions on the Riemannian space G/K.

(b) Suppose Σ is the restricted root system of a NCC symmetric space (in particular Σ is reduced). Let Π_0 be the basis of simple roots in Σ_0^+ and set $\Theta := \Pi_0$. Then φ_{Π_0} reduces to Unterberger's hypergeometric functions of the second type [Unt99] for arbitrary multiplicities, and to the spherical functions on NCC symmetric spaces for geometric multiplicities. In the latter case the Θ-spherical transform is the spherical Laplace transform of [FHÓ94] and [Ól97]. See also Chapter 8 in [HÓ96]. The Θ-spherical transform for $\Theta := \Pi_0$ and arbitrary non-geometric multiplicities does not appear in the literature.

The regularity properties of the Θ-spherical functions are reflected in the corresponding properties for the associated spherical transform of W_Θ-invariant spherical functions. We collect them in the following proposition.

Proposition 2.1.4

(a) *The Θ-spherical Fourier transform $\mathcal{F}_\Theta f(m;\lambda)$ of $f \in C_c(A_\Theta)^{W_\Theta}$ is a W_Θ-invariant meromorphic function of $\lambda \in \mathfrak{a}_\mathbb{C}^*$. Its singularities are located (counting multiplicities)*

in the polar set of the numerator function

$$n_\Theta^-(m;\lambda) := \prod_{\alpha\in\Sigma_{\mathrm{i}}^+\setminus\langle\Theta\rangle_{\mathrm{i}}^+} \Gamma\Big(-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}+\frac{1}{2}\Big)\Gamma\Big(-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}-\frac{m_{2\alpha}}{2}+1\Big).$$

When Σ is reduced, then all singularities are simple poles.

(b) *The normalized Θ-spherical Fourier transform $\mathcal{F}_\Theta^- f(m;\lambda)$ of $f \in C_c(A_\Theta)^{W_\Theta}$ is a W_Θ-invariant entire function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.*

(c) *Suppose Σ is reduced and m is an even multiplicity function. Set*

$$e_\Theta^-(m;\lambda) := \prod_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \prod_{k=-m_\alpha/2+1}^{m_\alpha/2-1} (\lambda_\alpha - k)\,.$$

Then $e_\Theta^-(m;\lambda)\mathcal{F}_\Theta f(m;\lambda)$ is a W_Θ-invariant entire function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

Proof. Immediate consequences of Theorems 1.5.11 and 1.6.6, respectively. □

Lemma 2.1.5 *There is a W_Θ-invariant tubular neighborhood U of A_Θ in $A_{\mathbb{C}}$ so that for all $(m,\lambda,a) \in \mathcal{M}^+ \times \mathfrak{a}_{\mathbb{C}}^* \times U$ the following equality of meromorphic functions holds:*

$$\varphi_\Pi(m;\lambda,a) = \sum_{w\in W_\Theta\setminus W} \frac{c_\Theta^{+,c}(m;w\lambda)}{c_\Theta^-(m;w\lambda)}\,\varphi_\Theta(m;w\lambda,a), \tag{2.5}$$

where the functions $c_\Theta^{+,c}(m;w\lambda)$ and $c_\Theta^-(m;w\lambda)$ are as in Definition 1.4.1.

Proof. This is an immediate consequence of the definition of the Θ-spherical functions and the W_Θ-invariance of $c_\Theta^{+,c}(m;\lambda)$, $c_\Theta^-(m;\lambda)$ and $\varphi_\Theta(m;\lambda,a)$. Indeed, since $c_\Pi^+(m;\lambda) = c_\Theta^+(m;\lambda)c_\Theta^{+,c}(m;\lambda)$, we obtain for all $m \in \mathcal{M}^+$, $a \in A^+$ and for generic $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$

$$\begin{aligned}
\varphi_\Pi(m;\lambda,a) &= \sum_{w\in W} c_\Pi^+(m;w\lambda)\Phi(m;w\lambda,a)\\
&= \sum_{w\in W_\Theta\setminus W}\sum_{w'\in W_\Theta} c_\Theta^+(m;w'w\lambda)c_\Theta^{+,c}(m;w'w\lambda)\Phi(m;w'w\lambda,a)\\
&= \sum_{w\in W_\Theta\setminus W} \frac{c_\Theta^{+,c}(m;w\lambda)}{c_\Theta^-(m;w\lambda)}\left(c_\Theta^-(m;w\lambda)\sum_{w'\in W_\Theta} c_\Theta^+(m;w'w\lambda)\Phi(m;w'w\lambda,a)\right)\\
&= \sum_{w\in W_\Theta\setminus W} \frac{c_\Theta^{+,c}(m;w\lambda)}{c_\Theta^-(m;w\lambda)}\varphi_\Theta(m;w\lambda,a).
\end{aligned}$$

The equality extends by analyticity to $(\lambda,a) \in \mathfrak{a}_{\mathbb{C}}^* \times U$ for a suitable W-invariant tubular neighborhood U of A_Θ in $A_{\mathbb{C}}$ on which $\varphi_\Pi(m;\lambda;a)$ and $\varphi_\Theta(m;w\lambda,a)$ (all $w \in W_\Theta/W$) are holomorphic functions of a. □

Example 2.1.6

(a) In the geometric case with $\Theta = \emptyset$, equation (2.5) reduces to the classical Harish-Chandra expansion of the spherical functions on G/K with respect to the Harish-Chandra series $\Phi(m;\lambda,a)$.

(b) When $(\mathfrak{a}, \Sigma, m)$ corresponds to a NCC symmetric space G/H, then (2.5) reduces to Ólafsson's functional relation between spherical functions on the Riemannian dual space G/K and spherical functions on G/H ([Ól97], Theorem 5.9).

Every $f \in C_c(A_\Theta)^{W_\Theta}$ (resp., $f \in C_c^\infty(A_\Theta)^{W_\Theta}$) extends uniquely to a W-invariant compactly supported continuous (resp., C^∞) function on A, which we still denote by f.

Corollary 2.1.7 *For every $f \in C_c(A_\Theta)^{W_\Theta}$ we have the following equality of meromorphic functions on $\mathfrak{a}_{\mathbb{C}}^*$:*

$$\mathcal{F}_{\text{II}} f(m;\lambda) = \sum_{w \in W_\Theta \backslash W} \frac{c_\Theta^{+,c}(m; w\lambda)}{c_\Theta^-(m; w\lambda)}\, \mathcal{F}_\Theta f(m; w\lambda), \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^*. \tag{2.6}$$

Proof. This follows immediately from Lemma 2.1.5. □

2.2 Estimates for the Harish-Chandra series

In this section we prove estimates for the Harish-Chandra series $\Phi(m;\lambda,a)$ which are uniform in $a \in A^+$ and locally uniform in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. For the multiplicity function m, we assume $m \in \mathcal{M}^+$, with the convention that $m_\alpha = 0$ is equivalent to $\alpha \notin \Sigma$.

Our methods are a modification of the celebrated estimates of Gangolli [Gan71]. As in [Gan71], we estimate the modified Harish-Chandra series

$$\Psi(m;\lambda,a) := \delta(m/2)\, \Phi(m;\lambda,a), \qquad a \in A^+, \lambda \in \mathfrak{a}_{\mathbb{C}}^*. \tag{2.7}$$

The functions Ψ has itself a series expansion of the form

$$\Psi(m;\lambda,a) = \sum_{\mu \in 2\Lambda} a_\mu(m;\lambda) e^{(\lambda-\mu)(\log a)} = e^{\lambda(\log a)} \sum_{\mu \in 2\Lambda} a_\mu(m;\lambda) e^{-\mu(\log a)}, \qquad a \in A^+,$$

which is obtained as Cauchy product of the series expansions of Φ and $\delta(m/2)$ (see below for the details). Since $\mu(\log a) \in \mathbb{R}$ for all $a \in A^+$ and $\mu \in 2\Lambda$, we have

$$|\Psi(m;\lambda,a)| \le e^{\Re\lambda(\log a)} \sum_{\mu \in 2\Lambda} |a_\mu(m;\lambda)|\, e^{-\mu(\log a)}.$$

An estimate for Ψ is therefore obtained when we can estimate the coefficients $a_\mu(m;\lambda)$. For this purpose, we first determine a differential equation satisfied by Ψ. Formal insertion of the series expansion for Ψ produces recurrence relations which we can employ inductively for estimating the $a_\mu(m;\lambda)$. Finally we go back and produce the required estimate for Φ using the defining equation (2.7).

The point of using Ψ instead of Φ is that Ψ satisfies a second order differential equation without first order terms; see equation (2.8) below. This simplifies the structure of the recurrence relations among the coefficients $a_\mu(m;\lambda)$.

The estimation procedure for the Harish-Chandra series just described has been first used by Gangolli, and has been inspired by the classical method of bringing the hypergeometric differential equation in Liouville's normal form (see [Gan71] and [Gan72] for further details).

Our final estimates differ from those in [Gan71] in two distinguishing points. First of all, we give more precise information on the behavior of the Harish-Chandra series on the walls of A^+. Secondly, Gangolli's estimates are uniform on the set of all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ satisfying $\Re\langle\lambda, \alpha\rangle \leq 0$ for all $\alpha \in \Sigma^+$. This set avoids the singularities of $\Phi(m; \lambda, a)$. Our applications in Section 2.3 require that we consider the entire $\mathfrak{a}_{\mathbb{C}}^*$. We are not able to deal with the entire $\mathfrak{a}_{\mathbb{C}}^*$ at one time. But we can move step by step in the region containing the singularities and provide uniform estimates on the sets

$$\mathfrak{a}_{\mathbb{C}}^*(R) := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \Re\langle\lambda, \alpha\rangle < R \quad \text{for all } \alpha \in \Sigma^+\},$$

for each $R > 0$. The λ-singularities of $\Phi(m; \lambda, a)$ are simple poles located on a locally finite family of complex hyperplanes and there are only finitely many singular hyperplanes intersecting $\mathfrak{a}_{\mathbb{C}}^*(R)$. Figure 2.2 exemplifies the situation in the rank-one case. The λ-singularities on $\mathfrak{a}_{\mathbb{C}}^*(R)$ are therefore cancelled by multiplication by a polynomial function p_R which is a finite product of linear factors in $\langle\lambda, \alpha\rangle$ with $\alpha \in \Sigma^+$ (see Lemmas 2.2.3 and 2.2.4 below for details). To carry out the two distinguishing points, we shall also employ ideas of Wallach [Wal90] and of van den Ban and Schlichtkrull [BS97].

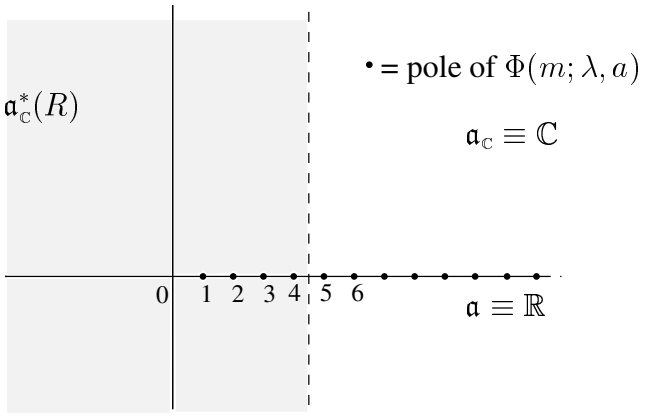


Figure 2.1: A set $\mathfrak{a}_{\mathbb{C}}^*(R)$, $R > 0$, in rank-one

We now enter into the details of the estimates. Recall that the Harish-Chandra series is a solution of the differential equation (1.8). By (2.7) and by Theorem 1.2.15, the function $\Psi(m; \lambda, a)$ satisfies therefore the differential equation

$$S(m)\Psi(m; \lambda, a) = \langle\lambda, \lambda\rangle\Psi(m; \lambda, a), \qquad a \in A^+, \tag{2.8}$$

where

$$S(m) := L_A + \sum_{\alpha \in \Sigma^+} \frac{m_\alpha(2 - m_\alpha - 2m_{2\alpha})}{(e^\alpha - e^{-\alpha})^2} \langle\alpha, \alpha\rangle$$

is the Schrödinger operator $S(m)$ of the generalized Calogero-Moser system associated with $(\mathfrak{a}, \Sigma, m)$. Let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ denote the fundamental system of simple roots in Σ^+. For any $T > 0$, we define

$$\mathfrak{a}^+(T) := \{H \in \mathfrak{a}^+ : \alpha_j(H) > T \quad \text{for all } j = 1, \ldots, l\} \tag{2.9}$$

and set $A^+(T) := \exp(\mathfrak{a}^+(T))$. The set $\mathfrak{a}^+(T)$ for the system A_2 is represented in Figure 2.2.

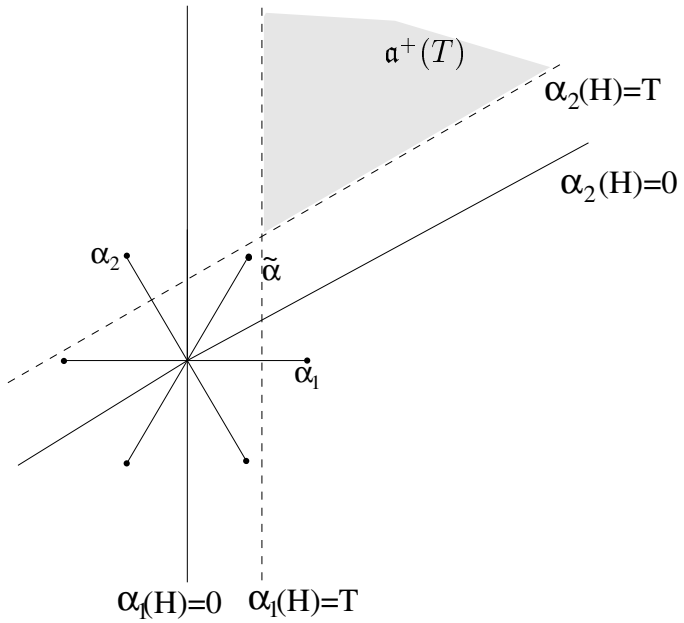


Figure 2.2: The set $\mathfrak{a}^+(T)$, $T > 0$, for the root system A_2

Recall the definition of Λ from Section 1.3. The function

$$\delta\,(m/2; a) = e^{\rho(m)(\log a)} \prod_{\alpha \in \Sigma^+} \left(1 - e^{-2\alpha(\log a)}\right)^{m_\alpha/2}$$

admits in A^+ the series expansion

$$\delta\,(m/2; a) = e^{\rho(m)(\log a)} \sum_{\mu \in 2\Lambda} b_\mu(m) e^{-\mu(\log a)} \tag{2.10}$$

with $b_0(m) = 1$. The series converges absolutely in A^+ and uniformly in $A^+(T)$ for all $T > 0$. These statements will be proven in a more general situation in Lemma 2.2.8 and Remark 2.2.9 below.

Recall also from (1.24) the notation $\Gamma_\eta(m; \lambda)$ for the coefficients of the Harish-Chandra series. Proposition 1.3.2 then implies that

$$\Psi(m; \lambda, a) = \sum_{\mu \in 2\Lambda} a_\mu(m; \lambda) e^{(\lambda - \mu)(\log a)}, \qquad a \in A^+,$$

where

$$a_\mu(m;\lambda) := \sum_{\substack{\nu,\eta\in 2\Lambda\\ \nu+\eta=\mu}} b_\nu(m)\Gamma_\eta(m;\lambda)$$

with $a_0(m;\lambda)=1$.

Let $T>0$ be arbitrarily fixed. The series for Ψ converges absolutely and uniformly on $A^+(T)$. Thus it can be differentiated term by term arbitrarily often. Let $\{H_j\}_{j=1}^l$ be an orthonormal basis in $\mathfrak{a}$ and set $\partial_j := \partial(H_j)$. Since

$$L_A e^{(\lambda-\mu)(\log a)} = \sum_{j=1}^l \partial_j^2 e^{(\lambda-\mu)(\log a)} = \sum_{j=1}^l (\lambda-\mu)(H_j)^2 e^{(\lambda-\mu)(\log a)} = \langle\lambda-\mu,\lambda-\mu\rangle e^{(\lambda-\mu)(\log a)},$$

we obtain

$$L_A\Psi(m;\lambda,a) = \sum_{\mu\in 2\Lambda}\langle\lambda-\mu,\lambda-\mu\rangle a_\mu(m;\lambda)e^{(\lambda-\mu)(\log a)}, \qquad a\in A^+(T), \tag{2.11}$$

Notice that

$$\frac{1}{(e^\alpha-e^{-\alpha})^2} = \sum_{k=1}^\infty k e^{-2k\alpha}.$$

Substitute (2.11) in the differential equation (2.8) and get

$$\sum_{\mu\in 2\Lambda} a_\mu(m;\lambda)\langle\lambda-\mu,\lambda-\mu\rangle e^{(\lambda-\mu)(\log a)} =$$
$$= \left[\sum_{\alpha\in\Sigma^+}\frac{m_\alpha(2-m_\alpha-2m_{2\alpha})}{(e^{\alpha(\log a)}-e^{-\alpha(\log a)})^2}\langle\alpha,\alpha\rangle + \langle\lambda,\lambda\rangle\right]\sum_{\mu\in 2\Lambda} a_\mu(m;\lambda)e^{(\lambda-\mu)(\log a)}$$

Since all series converge absolutely, we can multiply them out and collect the coefficients of the exponentials. This yields

$$\sum_{\mu\in 2\Lambda} a_\mu(m;\lambda)\langle\mu-2\lambda,\mu\rangle e^{(\lambda-\mu)(\log a)} =$$
$$= \sum_{\mu\in 2\Lambda}\sum_{\alpha\in\Sigma^+} m_\alpha(2-m_\alpha-2m_{2\alpha})\langle\alpha,\alpha\rangle \sum_{\substack{k\in\mathbb{N}\\ \mu-2k\alpha\in 2\Lambda}} k a_{\mu-2k\alpha}(m;\lambda)e^{(\lambda-\mu)(\log a)}.$$

Equating the coefficients of $e^{(\lambda-\mu)(\log a)}$ on both sides, we obtain the recursion relations

$$a_0(m;\lambda)=1,$$

and

$$a_\mu(m;\lambda)\langle\mu-2\lambda,\mu\rangle = \sum_{\alpha\in\Sigma^+} m_\alpha(2-m_\alpha-2m_{2\alpha})\langle\alpha,\alpha\rangle \sum_{\substack{k\in\mathbb{N}\\ \mu-2k\alpha\in 2\Lambda}} k a_{\mu-2k\alpha}(m;\lambda) \tag{2.12}$$

for $\mu\in 2\Lambda\setminus\{0\}$. They are a simplified version of the original relations (3.21) in [Gan71]. Our first goal is to use the recursion relations to determine growth estimates for the coefficients $a_\mu(m;\lambda)$.

We first need a lemma, which is a local version of a lemma in [Hör63]. We employ the usual multi-index notation: If $q = (q_1, \ldots, q_l) \in \mathbb{N}_0^l$ denotes a multi-index and $z = (z_1, \ldots, z_l)$ is an arbitrary point in $\mathbb{C}^l$, we write

$$\begin{aligned} |q| &:= \sum_{j=1}^{l} q_j, \\ z^q &:= z_1^{q_1} \cdots z_l^{q_l}, \\ \partial^q &:= \frac{\partial^{q_1}}{\partial z^{q_1}} \cdots \frac{\partial^{q_l}}{\partial z^{q_l}}. \end{aligned}$$

We moreover denote by $|z| := \left(\sum_{j=1}^{l} |z_j|^2\right)^{1/2}$ the norm of z associated with the natural Hermitian structure $(y, z) := \sum_{j=1}^{l} y_j \overline{z_j}$ in $\mathbb{C}^l$.

Lemma 2.2.1

(a) *Let p be a polynomial in the complex variable z of degree $\leq n$. Let $\varepsilon > 0$ be fixed. Suppose f is a holomorphic function on a domain $D \subset \mathbb{C}$ with $D \supset \{z \in \mathbb{C} : |z| \leq \varepsilon\}$. Finally, let $\psi : [0, \infty) \to \mathbb{R}$ be a non-negative integrable function with compact support $\operatorname{supp} \psi \subset [0, \varepsilon]$. Then for all $k = 0, 1, \ldots, n$ we have*

$$\left| f(0) \frac{d^k p}{dz^k}(0) \right| \int_{\mathbb{C}} |z|^k \, \psi(|z|) \, dz \leq C_{n,k} \int_{\mathbb{C}} |f(z)p(z)| \, \psi(|z|) \, dz,$$

where $C_{n,k} := \frac{n!}{(n-k)!}$. In the above integrals, dz denotes the Lebesgue measure on $\mathbb{C}$.

(b) *Let p be a polynomial in $z = (z_1, \ldots, z_l) \in \mathbb{C}^l$ of degree n_j in the variable z_j for all $j = 1, \ldots, l$. For a fixed $\varepsilon > 0$, let $r(\varepsilon) > 0$ be chosen so that*

$$P_{r(\varepsilon)} := \{z \in \mathbb{C}^l : |z_j| \leq r(\varepsilon) \text{ for all } j = 1, \ldots, l\} \subset \{z \in \mathbb{C}^l : |z| < \varepsilon\}.$$

Suppose F is a holomorphic function on a domain $D \subset \mathbb{C}^l$ with $D \supset \{z \in \mathbb{C}^l : |z| < \varepsilon\}$. Finally, let $\Psi : \mathbb{C}^l \to \mathbb{R}$ be a non-negative integrable function depending only on $|z_1|, \ldots, |z_l|$ and with compact support $\operatorname{supp} \Psi \subset P_{r(\varepsilon)}$. Then for every multi-index $q = (q_1, \ldots, q_l)$ with $q_j \leq n_j$ for all j, we have

$$|F(0) \, \partial^q p(0)| \int_{\mathbb{C}^l} |z^q| \, \Psi(|z|) \, dz \leq C_{n,q} \int_{\mathbb{C}^l} |F(z)p(z)| \, \Psi(|z|) \, dz,$$

Here $C_{n,q} := \prod_{j=1}^{l} C_{n_j, q_j}$ and $dz = dz_1 \ldots dz_n$ is the Lebesgue measure on $\mathbb{C}^l$.

(c) *Let p, n, q, F, ε and $r(\varepsilon)$ be as in part (b). Then there is a constant $C_{\varepsilon,n,q} > 0$ such that*

$$|F(0) \, \partial^q p(0)| \leq C_{\varepsilon,n,q} \int_{P_{r(\varepsilon)}} |F(z)p(z)| \, dz. \tag{2.13}$$

Proof. (cf. [Hör63], pp. 65–66.) For $\varepsilon = 1$, Lemma 3.1.3 in [Hör63] proves that for all $k = 0, 1, \ldots, n$ we have

$$2\pi \left| f(0) \frac{d^k p}{dz^k}(0) \right| \leq C_{n,k} \int_0^{2\pi} \left| f(e^{i\theta}) p(e^{i\theta}) \right| \, d\theta,$$

with $C_{n,k} := \frac{n!}{(n-k)!}$. Let $r \in (0, \varepsilon]$ be fixed, and apply this estimate to the holomorphic function $z \mapsto f(rz)$ and the polynomial $z \mapsto p(rz)$ to get

$$2\pi r^k \left| f(0) \frac{d^k p}{dz^k}(0) \right| \leq C_{n,k} \int_0^{2\pi} \left| f(re^{i\theta}) p(re^{i\theta}) \right| \, d\theta.$$

Multiplying both sides by $r\psi(r)$ and integrating over $[0, \varepsilon]$, we obtain part (a). Part (b) follows immediately from applications of part (a) on $P_{r(\varepsilon)}$ to each variable separately.

To prove (c), choose Ψ as the characteristic function of the set $P_{r(\varepsilon)}$ and choose the constant $C_{\varepsilon,n,q}$ as $C_{n,q} \left(\int_{\mathbb{C}} |z^q| \, \Psi(|z|) \, dz \right)^{-1}$. □

As a corollary we obtain a variation of a classical lemma by Malgrange (see also [Hel94], p. 288, and the references therein). It will play a decisive role at various stages.

Corollary 2.2.2 *Let $U \subset \mathbb{C}^l$ be any subset and let $\varepsilon > 0$. Consider the open ε-neighborhood*

$$U_\varepsilon := \{\zeta \in \mathbb{C}^l : \exists z \in U \quad \textit{with} \quad |\zeta - z| < \varepsilon\}.$$

Let H be a holomorphic function on U_ε, and let p be a polynomial such that $F := H/p$ is also holomorphic in U_ε. Let $w_1, \ldots, w_k$ be elements in $\mathbb{C}^l$ having all equal norm $|w_j| =: |w|$ $(j = 1, \ldots, k)$. Suppose there exists a constant $C > 0$ and $N \in \mathbb{N}_0$ such that

$$|H(z)| \leq C(1 + |z|)^N e^{\max_{j=1,\ldots,k}(\Re z, w_j)}$$

for all $z \in U_\varepsilon$. Then there is a constant $C_\varepsilon > 0$ such that

$$|F(z)| \leq C_\varepsilon (1 + |z|)^N e^{\max_{j=1,\ldots,k}(\Re z, w_j) + \varepsilon |w|}$$

for all $z \in U$.

Proof. Let $z \in U$ be fixed and apply (2.13) to the holomorphic function $\zeta \mapsto F(\zeta + z)$ and the polynomial $\zeta \mapsto p(\zeta + z)$, which are defined on an open domain containing $\{\zeta \in \mathbb{C}^l : |\zeta| < \varepsilon\}$. Then $\zeta + z \in U_\varepsilon$, on which the estimates for $H = Fp$ are given. Let $r(\varepsilon)$ be as in Lemma 2.2.1, and let $\left| P_{r(\varepsilon)} \right|$ denote the Lebesgue measure of $P_{r(\varepsilon)}$. Let n_j be the degree of p in the variable z_j. Since $r(\varepsilon) < \varepsilon$, we have for every multi-index $q = (q_1, \ldots, q_n)$ with

$q_j \leq n_j$ for all j

$$\begin{aligned}
|F(z)\,\partial^q p(z)| &\leq C_{\varepsilon,n,q} \int_{P_{r(\varepsilon)}} |F(\zeta+z)p(\zeta+z)|\; d\zeta \\
&\leq C_{\varepsilon,n,q} \left|P_{r(\varepsilon)}\right| \sup_{\zeta\in P_{r(\varepsilon)}} |F(\zeta+z)p(\zeta+z)| \\
&\leq C'_{\varepsilon,n,q} \sup_{|\zeta|<\varepsilon} |H(z+\zeta)| \\
&\leq C'_{\varepsilon,n,q} \sup_{|\zeta|<\varepsilon} (1+|z+\zeta|)^N e^{\max_{j=1,\dots,k}(\Re(z+\zeta),w_j)} \\
&\leq C'_{\varepsilon,n,q} \sup_{|\zeta|<\varepsilon} (1+|z+\zeta|)^N e^{\max_{j=1,\dots,k}(\Re z,w_j)+|\zeta||w|} \\
&\leq C'_{\varepsilon,n,q} (1+|z|+\varepsilon)^N e^{\max_{j=1,\dots,k}(\Re z,w_j)+\varepsilon|w|} \\
&\leq C''_{\varepsilon,n,q} (1+|z|)^N e^{\max_{j=1,\dots,k}(\Re z,w_j)+\varepsilon|w|}.
\end{aligned}$$

Since p is a polynomial, we can choose the multi-index q such that $\partial^q p(z)$ is a non-zero constant. This proves the required inequality on U. □

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we define $\Re\lambda, \Im\lambda \in \mathfrak{a}^*$ via

$$\Re\lambda(H) := \Re\big(\lambda(H)\big), \qquad \Im\lambda(H) := \Im\big(\lambda(H)\big)$$

for all $H \in \mathfrak{a}$. The assignment

$$|\lambda| := \langle\lambda,\lambda\rangle^{1/2} \tag{2.14}$$

defines a norm on $\mathfrak{a}^*$. We extend the norm $|\cdot|$ from $\mathfrak{a}^*$ to $\mathfrak{a}_{\mathbb{C}}^*$ by setting

$$|\lambda| = \left(|\Re\lambda|^2 + |\Im\lambda|^2\right)^{1/2}, \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^*. \tag{2.15}$$

Then $|\cdot|$ is the norm associated with the Hermitian inner product $(\lambda,\nu) := \langle\lambda,\overline{\nu}\rangle$, with $\overline{\nu} := \Re\nu - i\Im\nu$. By choosing an orthonormal basis, we can identify the Hermitian space $(\mathfrak{a}_{\mathbb{C}}^*, (\cdot,\cdot))$ with $\mathbb{C}^l$ together with its natural Hermitian structure. This justifies the application of Corollary 2.2.2 in our setting. Similarly, we define the norm $|H| := \langle H,H\rangle^{1/2}$ on $\mathfrak{a}$ and extend it to $\mathfrak{a}_{\mathbb{C}}$.

The fundamental system $\Pi = \{\alpha_1,\dots,\alpha_l\}$ is a basis for $\mathfrak{a}^*$. Define

$$\|\lambda\|_1 := \sum_{j=1}^{l} |\lambda_j|, \qquad \text{for } \lambda = \sum_{j=1}^{l} \lambda_j\alpha_j \in \mathfrak{a}^*. \tag{2.16}$$

Then $\|\mu\|_1 = \ell(\mu)$ is the level of μ for all $\mu \in \Lambda$. By equivalence of all norms in $\mathfrak{a}^*$, there is a constant $Q \geq 1$ such that for all $\lambda \in \mathfrak{a}^*$

$$Q^{-1}\|\lambda\|_1 \leq \langle\lambda,\lambda\rangle^{1/2} \leq Q\|\lambda\|_1.$$

In particular, for $\mu \in \Lambda$ we deduce that

$$Q^{-1}\ell(\mu) \leq |\mu| = \langle\mu,\mu\rangle^{1/2} \leq Q\ell(\mu). \tag{2.17}$$

The recursion relations (2.12) show that for $\mu \in 2\Lambda \setminus \{0\}$ each coefficient $a_\mu(m;\lambda)$ is a rational function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with at most simple poles along the hyperplanes

$$\mathcal{H}_\eta := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \langle \eta - 2\lambda, \eta\rangle = 0\}$$

for some $\eta \in 2\Lambda \setminus \{0\}$ with $\eta \leq \mu$.

In the following R will always denote a *finite* positive real number. For every $R > 0$ we set

$$\mathfrak{a}_{\mathbb{C}}^*(R) := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \Re\langle \lambda, \alpha\rangle < R \quad \text{for all } \alpha \in \Sigma^+\} \tag{2.18}$$

(see Figure 2.3).

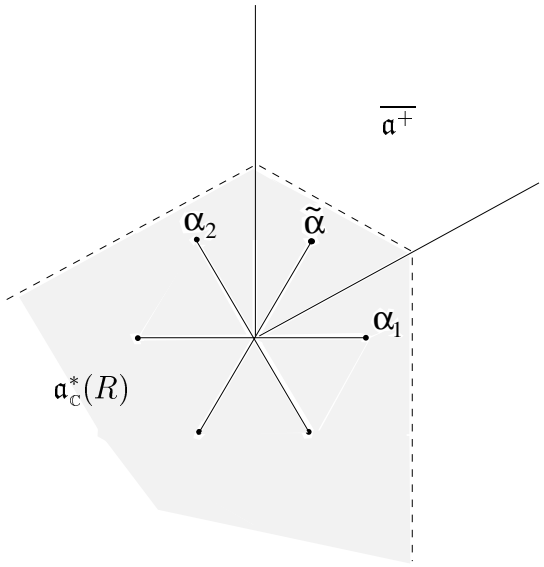


Figure 2.3: The real part of the set $\mathfrak{a}_{\mathbb{C}}^*(R)$, $R > 0$, for the root system A_2

Lemma 2.2.3 *Let $R > 0$ and $\eta \in 2\Lambda$, and let Q as in (2.17). If $\mathfrak{a}_{\mathbb{C}}^*(R) \cap \mathcal{H}_\eta \neq \emptyset$, then $\ell(\eta) < 2RQ^2$. Hence $\mathfrak{a}_{\mathbb{C}}^*(R)$ intersects at most finitely many hyperplanes $\mathcal{H}_\eta$ with $\eta \in 2\Lambda \setminus \{0\}$, and none if $R \leq 1/Q^2$.*

Proof. Let $\nu \in \mathfrak{a}_{\mathbb{C}}^*(R) \cap \mathcal{H}_\eta$, so $\langle \eta, \eta\rangle - 2\langle \nu, \eta\rangle = 0$. Write $\eta = \sum_{j=1}^l \eta_j \alpha_j$. Noticing that $\langle \eta, \eta\rangle \in \mathbb{R}$, we have

$$\langle \eta, \eta\rangle = \Re\langle \eta, \eta\rangle = 2\Re\langle \nu, \eta\rangle = 2\sum_j \eta_j \Re\langle \nu, \alpha_j\rangle < 2R\ell(\eta)$$

Thus $\ell(\eta)^2 \leq Q^2\langle \eta, \eta\rangle < 2RQ^2\ell(\eta)$, i.e. $\ell(\eta) < 2RQ^2$. The last statement in the lemma is a consequence of the fact that there are only finitely many $\eta \in 2\Lambda$ with $\ell(\eta) < 2RQ^2$ and that $\ell(\eta) \geq 2$ for $\eta \in 2\Lambda \setminus \{0\}$. Hence $2 \leq \ell(\eta) < 2RQ^2$ implies $R > 1/Q^2$. □

For $R > 0$, we set

$$\mathcal{X}_R := \{\eta \in 2\Lambda \setminus \{0\} : \mathcal{H}_\eta \cap \mathfrak{a}_{\mathbb{C}}^*(2R) \neq 0\}. \tag{2.19}$$

Then $\mathcal{X}_R$ is a finite set by Lemma 2.2.3, and it is empty for $R \leq 1/(2Q^2)$. We define the polynomial function p_R on $\mathfrak{a}_{\mathbb{C}}^*$ by

$$p_R(\lambda) := \prod_{\eta \in \mathcal{X}_R} \langle \eta - 2\lambda, \eta \rangle, \tag{2.20}$$

with the convention that $p_R(\lambda) \equiv 1$ when $\mathcal{X}_R = \emptyset$. We denote by $\deg p_R$ the degree of the polynomial p_R.

Lemma 2.2.4 *Suppose $R > 0$ and $\mu \in 2\Lambda$. Then:*

(a) *$p_R(\lambda) a_\mu(m;\lambda)$ is holomorphic in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(2R)$;*

(b) *For all $\nu < \mu$, the function $\dfrac{p_R(\lambda) a_\nu(m;\lambda)}{\langle \mu - 2\lambda, \mu \rangle}$ is holomorphic in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(2R)$.*

Proof. If $\mathcal{H}_\eta$ is a singular hyperplane for $a_\mu(m;\lambda)$ in $\mathfrak{a}_{\mathbb{C}}^*(2R)$, then $\mathcal{H}_\eta \cap \mathfrak{a}_{\mathbb{C}}^*(2R) \neq \emptyset$, hence $\eta \in \mathcal{X}_R$. The singularity along $\mathcal{H}_\eta$ is a simple pole, so it is removed by multiplication by $p_R(\lambda)$. This proves (a). For (b), observe first that, by the recursion relations, $\mathcal{H}_\mu$ is not a singular hyperplane for $a_\nu(m;\lambda)$ when $\nu < \mu$. The function $\frac{1}{\langle \mu - 2\lambda, \mu \rangle}$ is singular in $\mathfrak{a}_{\mathbb{C}}^*(2R)$ if and only if $\mathcal{H}_\mu \cap \mathfrak{a}_{\mathbb{C}}^*(2R) \neq \emptyset$. In this case $\langle \mu - 2\lambda, \mu \rangle$ divides $p_R(\lambda)$, so $\frac{p_R(\lambda) a_\nu(m;\lambda)}{\langle \mu - 2\lambda, \mu \rangle}$ is always holomorphic in $\mathfrak{a}_{\mathbb{C}}^*(2R)$. □

Lemma 2.2.5 *Suppose $R > 0$. Then there is $\varepsilon > 0$ with the property that*

$$\mathfrak{a}_{\mathbb{C}}^*(R)_\varepsilon := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \exists \nu \in \mathfrak{a}_{\mathbb{C}}^*(R) \quad \textit{with} \quad |\lambda - \nu| < \varepsilon\} \subset \mathfrak{a}_{\mathbb{C}}^*(2R).$$

Proof. Any $\varepsilon \in (0, R/\max_{\alpha \in \Sigma^+} |\alpha|]$ does the job. In fact, we have for such an ε, for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)_\varepsilon$ and $\alpha \in \Sigma^+$

$$\Re\langle \lambda - \nu, \alpha \rangle = \langle \Re(\lambda - \nu), \alpha \rangle \leq |\Re(\lambda - \nu)|\,|\alpha| < \varepsilon\,|\alpha| \leq R,$$

by Schwarz's inequality. Hence $\Re\langle \lambda, \alpha \rangle = \Re\langle \lambda - \nu, \alpha \rangle + \Re\langle \nu, \alpha \rangle < 2R$. □

Remark 2.2.6 Lemma 2.2.5 is the classical argument of "ε-neighborhoods" from the analysis. It is always more easily seen in a picture; cf. Figure 2.2 below.

Lemma 2.2.7 *Let $R' > R > 0$. Then there is a constant $C_{R,R',m} > 0$ with the following property: For every $\mu \in 2\Lambda$ with $\ell(\mu) \leq R'$*

$$|p_R(\lambda) a_\mu(m;\lambda)| \leq C_{R,R',m} (1 + |\lambda|)^{\deg p_R}$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$.

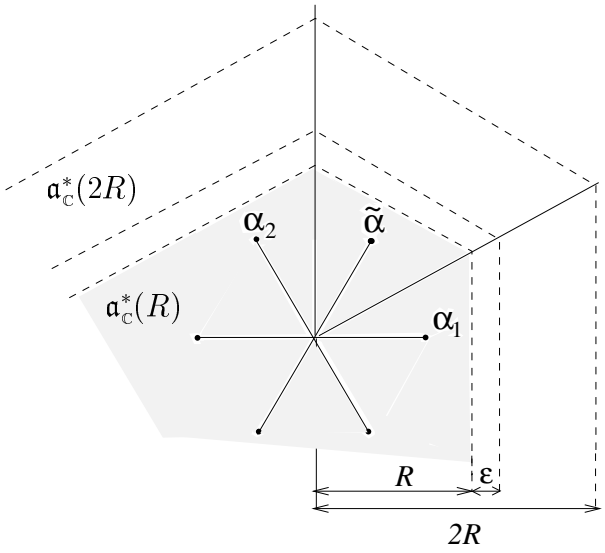


Figure 2.4: An ε-neighborhood of the set $\mathfrak{a}_{\mathbb{C}}^*(R)$, $R > 0$, for the root system A_2

Proof. Let $\varepsilon > 0$ be as in Lemma 2.2.5. For $\mu \in 2\Lambda \setminus \{0\}$ with $\ell(\mu) \leq R'$, we set

$$\varepsilon(\mu) := \varepsilon \cdot \frac{R' + 1 - \ell(\mu)}{R' + 1}.$$

Then $0 < \varepsilon(\mu) \leq \varepsilon$, hence

$$\mathfrak{a}_{\mathbb{C}}^*(R)_{\varepsilon(\mu)} \subset \mathfrak{a}_{\mathbb{C}}^*(R)_{\varepsilon} \subset \mathfrak{a}_{\mathbb{C}}^*(2R). \tag{2.21}$$

By induction on the level $\ell(\mu) \leq R'$, we prove the existence of a constant $C_\mu \geq 0$ such that

$$|p_R(\lambda) a_\mu(m;\lambda)| \leq C_\mu (1 + |\lambda|)^{\deg p_R}$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)_{\varepsilon(\mu)}$. The property is clearly true for $\mu = 0$. Assume it is true for all elements in 2Λ of level less than n ($< R'$) and prove it for $\mu \in 2\Lambda$ with $\ell(\mu) = n$. By the recursion relations (2.12), we have

$$|p_R(\lambda) a_\mu(m;\lambda)| \leq \sum_{\alpha \in \Sigma^+} m_\alpha \left| m_\alpha + 2m_{2\alpha} - 2 \right| \langle \alpha, \alpha \rangle \sum_{\substack{k \in \mathbb{N} \\ \mu - 2k\alpha \in 2\Lambda}} k \left| \frac{p_R(\lambda) a_{\mu - 2k\alpha}(m;\lambda)}{\langle \mu - 2\lambda, \mu \rangle} \right|.$$

Suppose $\mu - 2k\alpha \in 2\Lambda$ for some $\alpha \in \Sigma^+$ and $k \in \mathbb{N}$. By inductive hypothesis, there is a constant $C_{\mu - 2k\alpha} \geq 0$ so that

$$|p_R(\lambda) a_{\mu - 2k\alpha}(m;\lambda)| \leq C_{\mu - 2k\alpha} (1 + |\lambda|)^{\deg p_R}$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)_{\varepsilon(\mu - 2k\alpha)}$, where

$$\varepsilon(\mu - 2k\alpha) := \varepsilon \cdot \left(\frac{R' + 1 - \ell(\mu) + 2k\ell(\alpha)}{R' + 1} \right) \geq \varepsilon \cdot \left(\frac{R' - \ell(\mu) + 2}{R' + 1} \right) = \varepsilon(\mu) + \frac{\varepsilon}{R' + 1}.$$

Thus

$$\left[\mathfrak{a}_{\mathbb{C}}^*(R)_{\varepsilon(\mu)} \right]_{\varepsilon/(R'+1)} \subset \mathfrak{a}_{\mathbb{C}}^*(R)_{\varepsilon(\mu) + \varepsilon/(R'+1)} \subset \mathfrak{a}_{\mathbb{C}}^*(R)_{\varepsilon(\mu - 2k_\alpha)}.$$

By part (b) of Lemma 2.2.4, the function $\dfrac{p_R(\lambda)a_{\mu-2k\alpha}(m;\lambda)}{\langle\mu-2\lambda,\mu\rangle}$ is holomorphic in $\mathfrak{a}_{\mathbb{C}}^*(2R)$. Moreover, $\mathfrak{a}_{\mathbb{C}}^*(R)_{\varepsilon(\mu-2k_\alpha)} \subset \mathfrak{a}_{\mathbb{C}}^*(2R)$ by (2.21).

Corollary 2.2.2 implies then the existence of a constant $C'_{\mu-2k\alpha} \geq 0$ such that

$$\left|\frac{p_R(\lambda)a_{\mu-2k\alpha}(m;\lambda)}{\langle\mu-2\lambda,\mu\rangle}\right| \leq C'_{\mu-2k\alpha}(1+|\lambda|)^{\deg p_R}$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)_{\varepsilon(\mu)}$. The inductive step is therefore concluded when we set

$$C_\mu := \sum_{\alpha\in\Sigma^+} m_\alpha \left|m_\alpha + 2m_{2\alpha} - 2\right| \langle\alpha,\alpha\rangle \sum_{\substack{k\in\mathbb{N}\\ \mu-2k\alpha\in 2\Lambda}} kC'_{\mu-2k\alpha}.$$

This also proves the lemma, since there are only finitely many $\mu \in 2\Lambda$ satisfying $\ell(\mu) \leq R'$.

□

As in [Wal90], we now define for $c \in [0,\infty)$

$$\delta_c(m) := e^{c\rho(m)}\delta(m)^{-c/2} = e^{c\rho(m)}\delta(-cm/2) = \prod_{\alpha\in\Sigma^+}\left(1-e^{-2\alpha}\right)^{-cm_\alpha/2}. \tag{2.22}$$

The last equality is obtained by noticing that

$$\prod_{\alpha\in\Sigma^+}\left(e^{\alpha}-e^{-\alpha}\right)^{-cm_\alpha/2} = e^{-c\rho(m)}\prod_{\alpha\in\Sigma^+}\left(1-e^{-2\alpha}\right)^{-cm_\alpha/2}.$$

Observe also that $\delta_c \equiv 1$ for $c = 0$.

Lemma 2.2.8 *For every fixed $c \in \mathbb{R}$, we have the series expansion on A^+*

$$\prod_{\alpha\in\Sigma^+}\left(1-e^{-2\alpha}\right)^{-cm_\alpha/2} = \sum_{\mu\in 2\Lambda} d_\mu(m;c)e^{-\mu} \tag{2.23}$$

with $d_0(m;c) = 1$. The series converges absolutely in A^+ and uniformly in $A^+(T)$ for all $T > 0$.

If we assume $c \in (0,\infty)$, then $d_\mu(m;c) > 0$ for all $\mu \in 2\Lambda$.

Proof. For $r \in \mathbb{R}$ and $n \in \mathbb{N}_0$, we adopt the familiar notation

$$\binom{r}{n} := \prod_{h=1}^{n}\frac{r-h+1}{h}$$

for the binomial coefficients. As usual, empty products are defined to be 1. Recall that the binomial series

$$(1+x)^r = \sum_{k=0}^{\infty}\binom{r}{k}x^k$$

converges absolutely in $|x| < 1$ and uniformly in $|x| \leq S < 1$. Therefore for $\alpha \in \Sigma^+$ and $a \in A^+$, we have

$$\left(1-e^{-2\alpha(\log a)}\right)^{-cm_\alpha/2} = \sum_{k=0}^{\infty}\binom{-cm_\alpha/2}{k}(-1)^k e^{-2k\alpha(\log a)}, \tag{2.24}$$

with absolute convergence in A^+ and uniform convergence in $A^+(T)$ for all $T > 0$. Write $\Sigma^+ = \{\beta_1, \ldots, \beta_n\}$. We form the Cauchy product

$$\begin{aligned}
&\prod_{\alpha\in\Sigma^+} \left(1 - e^{-2\alpha(\log a)}\right)^{-cm_\alpha/2} \\
&\quad = \prod_{\alpha\in\Sigma^+} \sum_{k=0}^{\infty} \binom{-cm_\alpha/2}{k} (-1)^k e^{-2k\alpha(\log a)} \\
&\quad = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \binom{-cm_{\beta_1}/2}{k_1} \cdots \binom{-cm_{\beta_n}/2}{k_n} (-1)^{k_1+\cdots+k_n} e^{-2(k_1\beta_1+\cdots+k_n\beta_n)(\log a)} \\
&\quad = \sum_{\mu\in 2\Lambda} d_\mu(m;c) e^{-\mu(\log a)},
\end{aligned}$$

where

$$d_\mu(m;c) := \sum_{\substack{k_1,\ldots,k_n\in\mathbb{N}_0 \\ 2(k_1\beta_1+\cdots+k_n\beta_n)=\mu}} (-1)^{k_1+\cdots+k_n} \binom{-cm_{\beta_1}/2}{k_1} \cdots \binom{-cm_{\beta_n}/2}{k_n}.$$

As well as each of the series (2.24), also their Cauchy product converges absolutely in A^+ and uniformly in $A^+(T)$ for all $T > 0$. Observe that $d_0(m;c) = 1$.

If $x > 0$, then

$$(-1)^k \binom{-x}{k} := (-1)^k \prod_{h=1}^{k} \left(\frac{-x-h+1}{h}\right) = \prod_{h=1}^{k} \frac{x+h-1}{h} > 0.$$

Suppose then $c \in (0,\infty)$. Since we are assuming $m_\alpha > 0$ for all $\alpha \in \Sigma^+$, we have $(-1)^k \binom{-cm_\alpha/2}{k} > 0$ for all $\alpha \in \Sigma^+$ and $k \in \mathbb{N}_0$. In this case we therefore obtain $d_\mu(m;c) > 0$ for all $\mu \in 2\Lambda$. □

Remark 2.2.9 Lemma 2.2.8 with $c = -1$ justifies the statements for the series expansion (2.10) of $\delta(m/2)$. In this case, the coefficients $b_\mu(\lambda) := d_\mu(m;-1)$ need not be positive for $\mu \in 2\Lambda \setminus \{0\}$.

The idea to estimate the modified Harish-Chandra series $\Psi(m;\lambda,a)$ by means of the function $\delta_c(m)$ is the following. Suppose $c \in [0,\infty)$. According to Lemma 2.2.8, we have the series expansion on A^+

$$\delta_c(m) = \sum_{\mu\in 2\Lambda} d_\mu(m;c) e^{-\mu} \tag{2.25}$$

with $d_0(m;c) = 1$ and $d_\mu(m;c) \geq 0$ for all $\mu \in 2\Lambda$. Moreover $d_\mu(m;c) > 0$ for all $\mu \in 2\Lambda$ when $c \in (0,\infty)$. The series converges absolutely and uniformly on $A^+(T)$ for all $T > 0$. In particular it can be differentiated term by term. We first determine a differential equation satisfied by $\delta_c(m)$. Formally inserting in this differential equation the series (2.25), we obtain recursion relations for the coefficients $d_\mu(m;c)$. These recursion relations can be used to compare $|a_\mu(m,\lambda)|$ and $d_\mu(m;c)$ for the same $\mu \in 2\Lambda$. This provides an estimate for $\left|e^{-\lambda}\Psi(m;\lambda)\right|$ via $\delta_c(m)$. A suitable value of the parameter c is determined along the way.

Lemma 2.2.10 *For all $m \in \mathcal{M}^+$, $c \in [0,\infty)$ and $H \in \mathfrak{a}$, the function $\delta_c(m)$ satisfies the differential equation on A^+*

$$\begin{aligned}&\big(L_A - \partial(H)\big)\delta_c(m) = \\ &= c\sum_{\alpha\in\Sigma^+}\left[m_\alpha(\coth\alpha - 1)\left(\frac{\alpha(H)}{2} - c\langle\rho(m),\alpha\rangle\right) + \frac{m_\alpha(cm_\alpha+2cm_{2\alpha}+2)}{(e^\alpha - e^{-\alpha})^2}\langle\alpha,\alpha\rangle\right]\delta_c(m).\end{aligned}$$

Proof. Let $m \in \mathcal{M}^+$, $c \in [0,\infty)$ be fixed. For brevity, we omit the dependence on m in the following computations. Observe that for all $H \in \mathfrak{a}$

$$\begin{aligned}\delta^{c/2}\partial(H)\delta^{-c/2} &= -\frac{c}{2}\,\delta^{-1}\partial(H)\delta,\\ \delta^{-1}\partial(H)\delta &= \sum_{\alpha\in\Sigma^+} m_\alpha\alpha(H)\coth\alpha,\\ \partial(H)\left(\delta^{-1}\partial(H)\delta\right) &= -4\sum_{\alpha\in\Sigma^+} m_\alpha\alpha(H)^2\frac{1}{(e^\alpha - e^{-\alpha})^2}.\end{aligned}$$

Hence from

$$\begin{aligned}\delta_c^{-1}\partial(H)\delta_c &= c\rho(H) - \frac{c}{2}\,\delta^{-1}\partial(H)\delta,\\ \delta_c^{-1}\partial(H)^2\delta_c &= \left[c\rho(H) - \frac{c}{2}\,\delta^{-1}\partial(H)\delta\right]^2 - \frac{c}{2}\,\partial(H)\left(\delta^{-1}\partial(H)\delta\right),\end{aligned}$$

we obtain for $L_A = \sum_{j=1}^l \partial_j^2$

$$\begin{aligned}\delta_c^{-1}L_A(\delta_c) = c^2\langle\rho,\rho\rangle &+ \frac{c^2}{4}\sum_{\alpha,\beta\in\Sigma^+} m_\alpha m_\beta\langle\alpha,\beta\rangle\coth\alpha\coth\beta\\ &- c^2\sum_{\alpha\in\Sigma^+} m_\alpha\langle\rho,\alpha\rangle\coth\alpha + 2c\sum_{\alpha\in\Sigma^+} m_\alpha\langle\alpha,\alpha\rangle\frac{1}{(e^\alpha - e^{-\alpha})^2}.\end{aligned}$$

The equality

$$\frac{1}{4}\sum_{\alpha,\beta\in\Sigma^+} m_\alpha m_\beta\langle\alpha,\beta\rangle\coth\alpha\coth\beta = \langle\rho,\rho\rangle + \sum_{\alpha\in\Sigma^+}\frac{m_\alpha(m_\alpha+2m_{2\alpha})}{(e^\alpha - e^{-\alpha})^2}\langle\alpha,\alpha\rangle$$

is proven on p. 21 in [HS94]. Therefore, writing

$$\langle\rho,\rho\rangle = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha\langle\rho,\alpha\rangle,$$

we conclude that

$$\delta_c^{-1}L_A(\delta_c) = c^2\sum_{\alpha\in\Sigma^+} m_\alpha\langle\rho,\alpha\rangle(1-\coth\alpha) + c\sum_{\alpha\in\Sigma^+}\frac{m_\alpha(cm_\alpha+2cm_{2\alpha}+2)}{(e^\alpha - e^{-\alpha})^2}\langle\alpha,\alpha\rangle.$$

The claim then follows because for all $H \in \mathfrak{a}$ we have

$$\begin{aligned}-\delta_c^{-1}\partial(H)\delta_c &= -c\rho(H) + \frac{c}{2}\,\delta^{-1}\partial(H)\delta\\ &= \frac{c}{2}\sum_{\alpha\in\Sigma^+} m_\alpha(\coth\alpha - 1)\alpha(H).\end{aligned}$$

□

Lemma 2.2.11 *The coefficients $d_\mu(m;c)$ of the series (2.25) satisfy the recurrence relations*

$$d_0(m;c) = 1$$

and

$$\big(\langle\mu,\mu\rangle + \mu(H)\big)d_\mu(m;c) = \sum_{\alpha\in\Sigma^+}\sum_{\substack{k\in\mathbb{N}\\ \mu-2k\alpha\in 2\Lambda}}\Big[2cm_\alpha\left(\frac{\alpha(H)}{2} - c\langle\rho(m),\alpha\rangle\right) + \\ + kcm_\alpha(cm_\alpha + 2cm_{2\alpha} + 2)\langle\alpha,\alpha\rangle\Big]d_{\mu-2k\alpha}(m;c)$$

for $\mu \in 2\Lambda \setminus \{0\}$.

Proof. We formally insert the series (2.25) in the differential equation of Lemma 2.2.10. By termwise differentiation of the series (2.25) we obtain

$$\big(L_A - \partial(H)\big)(\delta_c) = \sum_{\mu\in\Lambda} d_\mu(m;c)\big(\langle\mu,\mu\rangle + \mu(H)\big)e^{-\mu}.$$

Recall that $(e^\alpha - e^{-\alpha})^{-2} = \sum_{k=1}^\infty ke^{-2k\alpha}$. Hence, by absolute convergence of the series,

$$\frac{1}{(e^\alpha - e^{-\alpha})^2}\delta_c = \sum_{\mu\in 2\Lambda}\sum_{\substack{k\in\mathbb{N}\\ \mu-2k\alpha\in 2\Lambda}} kd_{\mu-2k\alpha}(m;c)e^{-\mu}.$$

Finally, notice that

$$(\coth\alpha - 1)\delta_c = 2\sum_{\mu\in 2\Lambda}\sum_{\substack{k\in\mathbb{N}\\ \mu-2k\alpha\in 2\Lambda}} d_{\mu-2k\alpha}(m;c)e^{-\mu}.$$

The recursion relations are then obtained by equating the coefficients of $e^{-\mu}$ in the resulting equation. □

The following Lemma has been essentially proven in [Wal90], p. 782.

Lemma 2.2.12 *There exist constants $0 \le c < 1$ and $r > 1$ such that the inequality*

$$c(cm_\alpha + 2cm_{2\alpha} + 2) \ge r\,|m_\alpha + 2m_{2\alpha} - 2|$$

holds for all $\alpha \in \Sigma^+$.

Proof. Let $r > 1$ be arbitrarily fixed. For all $\alpha \in \Sigma^+$, define

$$c_{\alpha,r} := \frac{-1 + \sqrt{1 + r(m_\alpha + 2m_{2\alpha})\,|m_\alpha + 2m_{2\alpha} - 2|}}{m_\alpha + 2m_{2\alpha}}.$$

Notice that $c_{\alpha,r} \ge 0$. Set $c_r := \max_{\alpha\in\Sigma^+} c_{\alpha,r}$. Then $c_r \ge 0$. Moreover,

$$c_r^2(m_\alpha + 2m_{2\alpha}) + 2c_r - r\,|m_\alpha + 2m_{2\alpha} - 2| \ge 0 \tag{2.26}$$

for all $\alpha \in \Sigma^+$. Indeed $c_{\alpha,r}$ is the largest root of the quadratic equation $(m_\alpha + 2m_{2\alpha})x^2 + 2x - r\,|m_\alpha + 2m_{2\alpha} - 2| = 0$.

We now want to show that we can select $r > 1$ for which also $c_r < 1$ holds. Suppose $\beta \in \Sigma^+$ (depending on r) satisfies $c_r = c_{\beta,r}$. Then the inequality $c_r < 1$ can be written as

$$\frac{-1 + \sqrt{1 + r(m_\beta + m_{2\beta})\,|m_\beta + 2m_{2\beta} - 2|}}{m_\beta + 2m_{2\beta}} < 1.$$

It is satisfied precisely when r fulfills

$$1 + r(m_\beta + 2m_{2\beta})\,|m_\beta + 2m_{2\beta} - 2| < (m_\beta + 2m_{2\beta} + 1)^2. \tag{2.27}$$

Observe that

$$(m_\beta + 2m_{2\beta} + 1)^2 - 1 = (m_\beta + 2m_{2\beta})(m_\beta + 2m_{2\beta} + 2)$$

Therefore condition (2.27) is equivalent to

$$r\,|m_\beta + 2m_{2\beta} - 2| < m_\beta + 2m_{2\beta} + 2. \tag{2.28}$$

Since $|m_\alpha + 2m_{2\alpha} - 2| < m_\alpha + 2m_{2\alpha} + 2$ for all $\alpha \in \Sigma^+$, there exist r's so that

$$1 < r < \min_{\alpha \in \Sigma^+} \frac{m_\alpha + 2m_{2\alpha} + 2}{|m_\alpha + 2m_{2\alpha} - 2|},$$

and any of them satisfies (2.28). Let $\mathcal{R}$ denote the set of such r's. Then any c_r with $r \in \mathcal{R}$ satisfies all the requirements. □

Lemma 2.2.13 *Let $m \in \mathcal{M}^+$ be a multiplicity function. Then the system of inequalities of Lemma 2.2.12 admits the solution $c = 0$ if and only if $m_\alpha + 2m_{2\alpha} - 2 = 0$ for all $\alpha \in \Sigma^+$. In this case, $a_\mu(m;\lambda) = 0$ for all $\mu \in 2\Lambda \setminus \{0\}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.*

Proof. The first statement is obvious. Let $\mu \in 2\Lambda \setminus \{0\}$ be fixed. The recursion relations (2.12) define $a_\mu(m;\lambda)$ as a rational function of the variable $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Suppose $c = 0$. Then $m_\alpha + 2m_{2\alpha} - 2 = 0$ for all $\alpha \in \Sigma^+$, and the right-hand side of (2.12) vanishes. A rational function is identically zero when it vanishes on the complement of a lower dimensional manifold. The equality $a_\mu(m;\lambda)\langle \mu - 2\lambda, \mu\rangle = 0$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ therefore implies $a_\mu(m;\lambda) = 0$. □

Remark 2.2.14 Suppose m is geometric. Recall that we are assuming $m_\alpha > 0$ for $\alpha \in \Sigma$. If $m_{2\alpha} \neq 0$, then $2m_{2\alpha} - 2 \geq 0$. Hence the equality $m_\alpha + 2m_{2\alpha} - 2 = 0$ can be satisfied if and only if $2\alpha \notin \Sigma$ and $m_\alpha = 2$.

Fix $c \in [0, 1)$ and $r > 1$ satisfying the inequalities of Lemma 2.2.12. We choose $H \in \mathfrak{a}$ such that

$$\alpha(H) \geq \max\{2c\langle \rho(m), \alpha\rangle, 0\} \quad \text{for all } \alpha \in \Sigma^+. \tag{2.29}$$

Condition (2.29) ensures that the first summands in Lemma 2.2.11 are all positive and that $\mu(H) \geq 0$ for all $\mu \in 2\Lambda$.

Let $R > 0$. We wish to determine, for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$ and $\mu \in 2\Lambda$, an upper estimate of $|a_\mu(m;\lambda)|$ by means of the coefficients $d_\mu(m;c)$ of the series expansion of δ_c. For finitely many $\mu \in 2\Lambda$ this can be easily done by means of Lemma 2.2.7.

Lemma 2.2.15 *Let $c \in [0,1)$ and $r > 1$ satisfy the inequalities of Lemma 2.2.12. Suppose $R' > R > 0$. Then there is a constant $C_{R,R',m,c,r} > 0$ with the following property: For every $\mu \in 2\Lambda$ with $\ell(\mu) \leq R'$*

$$|p_R(\lambda) a_\mu(m;\lambda)| \leq C_{R,R',m,c,r}\; d_\mu(m;c)(1+|\lambda|)^{\deg p_R}$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$.

Proof. If $c = 0$, then $d_\mu(m;c) = 0$ for all $\mu \in 2\Lambda \setminus \{0\}$. But then also $a_\mu(m;\lambda) = 0$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $\mu \in 2\Lambda \setminus \{0\}$ by Lemma 2.2.13. The inequality is therefore trivially satisfied.

Suppose now that $c \in (0,1)$. Lemma 2.2.8 ensures then that $d_\mu(m;c) > 0$ for all $\mu \in 2\Lambda \setminus \{0\}$. Set

$$d := \min\{d_\mu(m;c) : \mu \in 2\Lambda, \ell(\mu) \leq R'\}.$$

Then $d > 0$ and the required inequality is satisfied with $C_{R,R',m,c,r} := C_{R,R',m}/d$, where $C_{R,R',m}$ is the constant of Lemma 2.2.7. □

To extend the inequalities of Lemma 2.2.15 to all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and *all* $\mu \in 2\Lambda$, we compare the recursion relations of $a_\mu(m;\lambda)$ and $d_\mu(m;c)$; see (2.12) and Lemma 2.2.11, respectively. Looking at these recursion relations, we realize that, when H is chosen as in (2.29), we need an inequality of the form

$$|\langle \mu,\mu\rangle - 2\langle \lambda,\mu\rangle| > \frac{1}{r}\left[\langle \mu,\mu\rangle + \mu(H)\right]$$

to hold for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$ and all $\mu \in 2\Lambda$. Such an inequality is clearly not possible because its right-hand side is always positive, whereas its left-hand side may be equal to zero. Nevertheless, the next lemma states that this inequality is satisfied by all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$ and by all $\mu \in 2\Lambda \setminus D_{R,r}$, where $D_{R,r}$ contains only finitely many elements. This will be enough, because for finitely many $\mu \in 2\Lambda$ we can apply Lemma 2.2.15.

Lemma 2.2.16 *Let $H \in \mathfrak{a}$ be chosen as in (2.29). For $r > 1$ and $R > 0$ we set*

$$D_{R,r} := \{\mu \in 2\Lambda : \langle \mu,\mu\rangle + \mu(H) \geq r\left[\langle \mu,\mu\rangle - 2R\ell(\mu)\right]\}.$$

Then $D_{R,r}$ is a finite set. More precisely,

$$D_{R,r} \subset \{\mu \in 2\Lambda : \ell(\mu) \leq R_r\}$$

with

$$R_r := \frac{Q}{r-1}\left(|H| + 2rRQ\right) \tag{2.30}$$

and Q as in (2.17). Furthermore,

$$|\langle \mu,\mu\rangle - 2\langle \lambda,\mu\rangle| > \frac{1}{r}\left[\langle \mu,\mu\rangle + \mu(H)\right].$$

for all $\mu \in 2\Lambda \setminus D_{R,r}$ and all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$.

Proof. With Q as in (2.17) and $\mu \in D_{R,r}$, we have

$$(r-1)\langle \mu, \mu \rangle \leq \mu(H) + 2rR\ell(\mu) \leq |\mu|\,|H| + 2rRQ\,|\mu|\,.$$

Hence $(r-1)\,|\mu| \leq |H| + 2rRQ$, which implies

$$\ell(\mu) \leq Q\,|\mu| \leq \frac{Q}{r-1}\,(|H| + 2rRQ) =: R_r.$$

Notice that R_r is finite because $r > 1$. Thus $D_{R,r} \subset \{\mu \in 2\Lambda : \ell(\mu) \leq R_r\}$ is a finite set.

Suppose $\mu \in 2\Lambda \setminus D_{R,r}$. Then

$$\langle \mu, \mu \rangle + \mu(H) < r\,[\langle \mu, \mu \rangle - 2R\ell(\mu)]\,.$$

This implies in particular that $\langle \mu, \mu \rangle - 2R\ell(\mu) > 0$ because of (2.29) and because $\langle \mu, \mu \rangle > 0$ when $\ell(\mu) > 0$. Write $\mu = \sum_{j=1}^{l} \mu_j \alpha_j$ with $\mu_j \in 2\mathbb{N}_0$. Then for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$ we have

$$\begin{aligned}
|\langle \mu, \mu \rangle - 2\langle \lambda, \mu \rangle| &\geq \Re\,[\langle \mu, \mu \rangle - 2\langle \lambda, \mu \rangle] \\
&= \langle \mu, \mu \rangle - 2\sum_j \mu_j \Re\langle \lambda, \alpha_j \rangle \\
&> \langle \mu, \mu \rangle - 2R\big(\sum_j \mu_j\big) \\
&= \langle \mu, \mu \rangle - 2R\ell(\mu) \\
&> \frac{1}{r}\,[\langle \mu, \mu \rangle + \mu(H)]\,.
\end{aligned}$$

□

Lemma 2.2.17 *Let $R > 0$ and let p_R be the polynomial defined in (2.20). Suppose $c \in [0, 1)$ is chosen as in Lemma 2.2.12 for some $r > 1$. Then there is a constant $C_{R,c,m} > 0$ such that*

$$|p_R(\lambda) a_\mu(m; \lambda)| \leq C_{R,c,m} d_\mu(m; c)(1 + |\lambda|)^{\deg p_R}$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$ and all $\mu \in 2\Lambda$.

Proof. We proceed by induction on $\ell(\mu)$. For a fixed R, let R_r be as in (2.30). Observe that $R_r > R$. Since the set of $\mu \in 2\Lambda$ with $\ell(\mu) \leq R_r$ is finite, the claim is true for all such μ by Lemma 2.2.15. Assume the claim as true for all $\eta \in 2\Lambda$ so that $\ell(\eta) < n$, and prove it for $\mu \in 2\Lambda$ with $\ell(\mu) = n$ (and we can assume $n > R_r$). Since $\ell(\mu) > R_r$, then $\mu \notin D_{R,r}$. Hence

$$\frac{1}{|\langle \mu, \mu \rangle - 2\langle \lambda, \mu \rangle|} < \frac{r}{\langle \mu, \mu \rangle + \mu(H)}.$$

by Lemma 2.2.16. From the recursion relations (2.12), the inductive hypothesis, Lemma

2.2.12 and (2.29), we obtain

$$
\begin{aligned}
&|p_R(\lambda)a_\mu(m;\lambda)| \leq \\
&\leq \frac{r}{\langle\mu,\mu\rangle+\mu(H)} \sum_{\alpha\in\Sigma^+} m_\alpha \,|m_\alpha+2m_{2\alpha}-2|\,\langle\alpha,\alpha\rangle \sum_{\substack{k\in\mathbb{N}\\ \mu-2k\alpha\in 2\Lambda}} k\,|p_R(\lambda)a_{\mu-2k\alpha}(m;\lambda)| \\
&\leq \frac{C_{R,c,m}}{\langle\mu,\mu\rangle+\mu(H)}\Big[\sum_{\alpha\in\Sigma^+} m_\alpha(r\,|2-m_\alpha-2m_{2\alpha}|)\langle\alpha,\alpha\rangle \sum_{\substack{k\in\mathbb{N}\\ \mu-2k\alpha\in 2\Lambda}} k d_{\mu-2k\alpha}(m;c)\Big](1+|\lambda|)^{\deg p_R} \\
&\leq \frac{C_{R,c,m}}{\langle\mu,\mu\rangle+\mu(H)}\Big[\sum_{\alpha\in\Sigma^+} cm_\alpha(cm_\alpha+2m_{2\alpha}+2)\langle\alpha,\alpha\rangle \sum_{\substack{k\in\mathbb{N}\\ \mu-2k\alpha\in 2\Lambda}} k d_{\mu-2k\alpha}(m;c)\Big](1+|\lambda|)^{\deg p_R} \\
&\leq \frac{C_{R,c,m}}{\langle\mu,\mu\rangle+\mu(H)}\Big\{\sum_{\alpha\in\Sigma^+}\sum_{\substack{k\in\mathbb{N}\\ \mu-2k\alpha\in 2\Lambda}} \Big[2cm_\alpha\Big(\frac{\alpha(H)}{2}-c\langle\rho(m),\alpha\rangle\Big)+ \\
&\qquad + kcm_\alpha(cm_\alpha+2cm_{2\alpha}+2)\langle\alpha,\alpha\rangle\Big] d_{\mu-2k\alpha}(m;c)\Big\}(1+|\lambda|)^{\deg p_R} \\
&= C_{R,c,m} d_\mu(m;c)(1+|\lambda|)^{\deg p_R}.
\end{aligned}
$$

The last equality is provided by the recursion relations for $d_\mu(m;c)$ in Lemma 2.2.11. This completes the inductive step. □

Theorem 2.2.18 *Let $R > 0$ and let p_R be the polynomial defined in (2.20). Suppose $m \in \mathcal{M}^+$. Then there exist $c \in [0,1)$ (depending only on the multiplicity function m) and $C_{R,c,m} > 0$ such that*

$$|p_R(\lambda)\Psi(m;\lambda,a)| \leq C_{R,c,m}\delta(m;a)^{-c/2}(1+|\lambda|)^{\deg p_R} e^{(c\rho(m)+\Re\lambda)(\log a)} \tag{2.31}$$

and

$$\big|p_R(\lambda)\delta(m;a)^{(c+1)/2}\Phi(m;\lambda,a)\big| \leq C_{R,c,m}(1+|\lambda|)^{\deg p_R} e^{(c\rho(m)+\Re\lambda)(\log a)} \tag{2.32}$$

for all $a \in A^+$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$.

We can choose $p_R \equiv 1$ for all sufficiently small $R > 0$ (e.g. $R \leq 1/(2Q^2)$ with Q as in (2.17)).

Proof. Because of Lemma 2.2.17

$$
\begin{aligned}
|p_R(\lambda)\Psi(m;\lambda,a)| &\leq \sum_{\mu\in 2\Lambda} |p_R(\lambda)a_\mu(m;\lambda)|\,\big|e^{(\lambda-\mu)(\log a)}\big| \\
&\leq C_{R,c,m}\sum_{\mu\in 2\Lambda} d_\mu(m;c)e^{-\mu(\log a)}(1+|\lambda|)^{\deg p_R} e^{\Re\lambda(\log a)} \\
&\leq C_{R,c,m}\delta_c(m;a)(1+|\lambda|)^{\deg p_R} e^{\Re\lambda(\log a)}.
\end{aligned}
$$

The first inequality in (a) therefore follows from the definition of δ_c (see 2.22), and the second from the equality $\Psi(m;\lambda,a) = \delta(m;a)^{1/2}\Phi(m;\lambda,a)$. □

Remark 2.2.19

(a) Two features of the estimates (2.32) will be important in the following discussion of the Θ-spherical transform:

(i) The condition $c \in [0,1)$ implies $(c+1)/2 \in [1/2,1)$;

(ii) Each $\mathfrak{a}_{\mathbb{C}}^*(R)$ contains a W_Θ-invariant open subset $\mathfrak{a}_{\mathbb{C},\Theta}^*(R)$ such that the set of the $\mathfrak{a}_{\mathbb{C},\Theta}^*(R)$ for $R > 0$ covers $\mathfrak{a}_{\mathbb{C}}^*$.

These are the properties that will allow us to extend the definition of the Θ-spherical Fourier transform also to functions which are not compactly supported on A_Θ.

(b) For $0 < R < 1/(2Q^2)$ we have $p_R(\lambda) = 1$ for all λ. Hence the estimate (2.32) reduces to

$$\delta(m;a)^{(c+1)/2}\,|\Phi(m;\lambda,a)| \le C_{R,c,m} e^{(c\rho(m)+\Re\lambda)(\log a)} \tag{2.33}$$

for all $a \in A^+$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$. For $R = 0$, the estimate (2.33) has been established in the geometric case by Wallach [Wal90]. Our optimal (=smallest) value for c coincides with the one determined in [Wal90] despite the use of different reductions for the recursion relations. The reduction methods used in Wallach's paper cannot be generalized to non-geometric multiplicities because they are based on Araki's tables of multiplicities.

Estimates of the type (2.32) have been determined in the more general case of Eisenstein integrals by van den Ban and Schlichtkrull in [BS97], where the exponent of δ is however left undetermined (it is also not stated whether it is smaller than 1).

(c) We introduce the notation $A^+ \ni a \to +\infty$ to indicate that $a \in A^+$ and that $\alpha(\log a) \to +\infty$ for all $\alpha \in \Sigma^+$. On the Lie algebra level, this means that $\log a \in \mathfrak{a}^+$ moves towards infinity in such a way that also its distance from *each* of the walls of the Weyl chamber $\mathfrak{a}^+$ goes to infinity. Since for $A^+ \ni a \to +\infty$

$$\delta(m;a)^{(c+1)/2} = e^{(c+1)\rho(m)(\log a)} \prod_{\alpha\in\Sigma^+} (1-e^{-2\alpha(\log a)})^{(c+1)m_\alpha/2} \sim e^{(c+1)\rho(m)(\log a)}$$

$$\Phi(m;\lambda,a) \sim e^{(\lambda-\rho(m))(\log a)},$$

we conclude that, for a certain constant $C_{\lambda,R} > 0$,

$$\left|p_R(\lambda)\delta(m;a)^{(c+1)/2}\Phi(m;\lambda,a)\right| \sim C_{\lambda,R}\, e^{(c\rho(m)+\Re\lambda)(\log a)}.$$

This shows that the estimate (2.32) is sharp for $A^+ \ni a \to +\infty$.

(d) For all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ the factor $\delta(m;a)^{(c+1)/2}$ cancels the possible singularities of $\Phi(m;\lambda,a)$ along the walls of A^+. The exact asymptotic behavior of $\Phi(m;\lambda,a)$ along the walls is in general not known. In the complex and rank-one cases, these asymptotics can be deduced by the explicit formulas for the Harish-Chandra series. We shall compare these exact asymptotics with our estimate in Examples 2.2.20 and 2.2.21 below.

(e) For $R > (1/2Q^2)$ the exponent $\deg p_R$ does not give a sharp estimate in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$. The first a-priori reason comes from Theorem 1.3.4, which states that the λ-singularities of $\Phi(m;\lambda,a)$ are much fewer than the singularities suggested by the recursion relations. The complex case (see Example 2.2.20 below) exemplifies this problem in its worst appearance. Furthermore, the estimates on $\mathfrak{a}_{\mathbb{C}}^*(R)$ have been obtained by means of p_R, which is the polynomial canceling all first-order singularities along the hyperplanes $\mathcal{H}_\eta$ intersecting the larger domain $\mathfrak{a}_{\mathbb{C}}^*(2R)$. We couldn't have replaced p_R by the polynomial $p_{R/2}$ (which is the one canceling the λ-singularities along the hyperplanes intersecting $\mathfrak{a}_{\mathbb{C}}^*(R)$) because of the application of Corollary 2.2.2 in Lemma 2.2.7. Indeed, Corollary 2.2.2 requires holomorphy on some ε-neighborhood of $\mathfrak{a}_{\mathbb{C}}^*(R)$. Both for our applications in section 2.4 and for possible applications in Paley-Wiener type theorems, the important feature of the estimate in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$ is its form $(1+|\lambda|)^{d(R)}e^{\Re\lambda}$, and not the exact value of the (non-negative) exponent $d(R)$. Since this form can be achieved in any case, our choice of working on $\mathfrak{a}_{\mathbb{C}}^*(2R)$ (rather than on $\mathfrak{a}_{\mathbb{C}}^*(R+\varepsilon)$ for some $\varepsilon > 0$) has only been dictated by the attempt of simplifying the notation.

Example 2.2.20 (The complex case) If Σ is reduced and $m_\alpha = 2$ for all $\alpha \in \Sigma$, then the right-hand side of the inequality in Lemma 2.2.12 vanishes. Thus the optimal (=smallest) value of c is in this case $c = 0$ (obtained for any arbitrary $r > 1$). The explicit form of the Harish-Chandra series $\Phi(2;\lambda,a) = \Delta(a)^{-1}e^{\lambda(\log a)}$ (see Example 1.3.11) shows that we can choose $p_R \equiv 1$ for all $R > 0$. Since $\delta(2)^{1/2} = \delta(1) = \Delta$, the estimate (2.32) reduces with $c = 0$ to

$$\Delta(a)\,|\Phi(2;\lambda,a)| \le C_R(1+|\lambda|)^{\deg p_R}e^{\Re\lambda(\log a)}$$

for all $a \in A^+$. For $R > 0$ small enough, Theorem 2.2.18 yields $\deg p_R = 0$. In this case the estimate is the best possible. For arbitrary $R > 0$ the values of $\deg p_R$ found via Theorem 2.2.18 are positive, and the term $(1+|\lambda|)^{\deg p_R}$ is in fact superfluous.

Example 2.2.21 (The rank-one case) In the rank-one case, the system of inequalities in Lemma 2.2.12 consists of only one inequality when $2\alpha \notin \Sigma^+$, and of two when $2\alpha \in \Sigma^+$. We now determine the minimal c_0 over all possible $c \in [0,1)$ which can be used in the estimate (2.32) for $\Phi(m;\lambda,t)$. Recall that in this chapter we convene that $m_{2\alpha} = 0$ is equivalent to $2\alpha \notin \Sigma$. Using the notation introduced in the proof of Lemma 2.2.12, we have

$$\inf_{r>1} c_{\alpha,r} = \frac{-1+\sqrt{1+(m_\alpha+2m_{2\alpha})\,|m_\alpha+2m_{2\alpha}-2|}}{m_\alpha+2m_{2\alpha}},$$
$$\inf_{r>1} c_{2\alpha,r} = \frac{-1+\sqrt{1+m_{2\alpha}\,|m_{2\alpha}-2|}}{m_{2\alpha}}$$

and

$$c_0 = \max\left\{\inf_{r>1} c_{\alpha,r}, \inf_{r>1} c_{2\alpha,r}\right\}.$$

Suppose that either Σ is reduced and $m_\alpha \geq 2$, or Σ is not reduced and $m_{2\alpha} \geq 2$. In both cases $m_\alpha + 2m_{2\alpha} - 2 \geq 0$, and in the non-reduced case we also have $m_{2\alpha} - 2 \geq 0$. Hence

$$\begin{aligned}\inf_{r>1} c_{\alpha,r} &= \frac{-1+\sqrt{(m_\alpha+2m_{2\alpha}-1)^2}}{m_\alpha+2m_{2\alpha}} = 1 - \frac{2}{m_\alpha+2m_{2\alpha}}\\ \inf_{r>1} c_{2\alpha,r} &= \frac{-1+\sqrt{(m_{2\alpha}-1)^2}}{m_{2\alpha}} = 1 - \frac{2}{m_{2\alpha}},\\ c_0 &= 1 - \frac{2}{m_\alpha+2m_{2\alpha}} \in [0,1).\end{aligned}$$

Thus the minimal exponent for $\delta(m)$ in the estimates (2.32) is

$$\frac{1+c_0}{2} = 1 - \frac{1}{m_\alpha+2m_{2\alpha}} \in [1/2, 1). \tag{2.34}$$

Notice that $c_0 = 0$ if and only if $m_\alpha + 2m_{2\alpha} = 2$. Because of the restrictions we are now imposing on m_α and $m_{2\alpha}$, this is equivalent to $m_\alpha = 2$ and $m_{2\alpha} = 0$. Therefore $(1+c_0)/2 = 1/2$ if and only if m corresponds to the complex rank-one case. Observe also that $(1+c_0)/2$ is always < 1, but, as formula (2.34) shows, it can be arbitrarily close to 1.

The wall of $A^+ \equiv (0,\infty)$ reduces in the rank-one case to the point $t = 0$. We know already from Remark 2.2.19 (c) that the estimate (2.32) is sharp for $t \to +\infty$. It is therefore enough to compare this estimate near $t = 0$ with the asymptotic behavior for $t \to 0^+$ deduced by the explicit formula for $\Phi(m;\lambda,t)$. We have as $t \to 0^+$

$$\delta(m;t)^{(1+c_0)/2} = \left[(2\sinh t)^{m_\alpha}(2\sinh(2t))^{m_{2\alpha}}\right]^{(1+c_0)/2} \sim C_m t^{m_\alpha+m_{2\alpha}-\left(\frac{m_\alpha+m_{2\alpha}}{m_\alpha+2m_{2\alpha}}\right)},$$

where C_m denotes a positive constant depending only on the multiplicity function m. Consequently (for some $C_{m,\lambda} > 0$)

$$\delta(m;t)^{-(1+c_0)/2} e^{(c_0\rho+\Re\lambda)t} \sim C_{m,\lambda} t^{\left(\frac{m_\alpha+m_{2\alpha}}{m_\alpha+2m_{2\alpha}}\right)-(m_\alpha+m_{2\alpha})}, \qquad t \to 0^+. \tag{2.35}$$

Recall from Example 1.3.10 that

$$\Phi(m;\lambda,t) = (2\sinh t)^{\lambda-\rho}\, {}_2F_1\left(\frac{\rho-\lambda}{2}, \frac{-m_\alpha/2+1-\lambda}{2}; 1-\lambda; -\sinh^{-2} t\right).$$

Since $-\sinh^{-2} t$ tends to $-\infty$ as $t \to 0^+$, the asymptotic behavior of the function $\Phi(m;\lambda,t)$ for $t \to 0^+$ can be deduced from the formulas for the analytic continuation of the hypergeometric function in a neighborhood of ∞. More precisely, consider the formula

$$\begin{aligned}\frac{{}_2F_1(a,b;c;z)}{\Gamma(c)} = {}& \frac{\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\, {}_2F_1\left(a, 1-c+a; 1-b+a; z^{-1}\right)\\ &+ \frac{\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\, {}_2F_1\left(b, 1-c+b; 1-a+b; z^{-1}\right),\end{aligned}$$

which is valid for $a - b \notin \mathbb{Z}$ and for $|\arg(-z)| < \pi$ (see [E$^+$53], 2.1.4 (17)). Since ${}_2F_1(\cdot,\cdot;\cdot;0) = 1$, we obtain that for all multiplicity functions m with $m_\alpha + m_{2\alpha}$ different from an odd integer and for some constant $C_{m,\lambda} > 0$

$$\frac{\Phi(m;\lambda,t)}{\Gamma(1-\lambda)} \sim C_{m,\lambda} t^{1-(m_\alpha+m_{2\alpha})}, \qquad t \to 0^+. \tag{2.36}$$

When $m_\alpha + m_{2\alpha}$ is an odd integer, we can apply the analytic continuation formula (see [E$^+$53] 2.1.4(18))

$$\frac{\Gamma(b+q)}{\Gamma(c)}\,{}_2F_1\,(b+q,b;c;z) = \frac{(-z)^{-b-q}}{\Gamma(c-b)}\sum_{n=0}^{\infty}\frac{(b)_{n+q}(1-c+b)_{n+q}}{n!(n+q)!}\,z^{-n}\,[\log(-z)+h_n] + (-z)^{-b}\sum_{n=0}^{q-1}\frac{(b)_n\Gamma(q-n)}{\Gamma(c-b-n)n!}\,z^{-n}, \quad (2.37)$$

which holds for $b \notin -\mathbb{N}_0$, $q \in \mathbb{N}_0$ and $|\arg(-z)| < \pi$. The summation $\sum_{n=0}^{q-1}$ means 0 if $q = 0$. Moreover, $(b)_n := \Gamma(b+n)/\Gamma(b)$ is the *shifted factorial*. Finally, h_n denotes a certain function of b, c, m, n (independent of z). In our situation, $q = (m_\alpha + m_{2\alpha} - 1)/2 \neq 0$, hence the dominant term for $t \to 0^+$ is the first summand in the second sum. This yields again

$$\frac{\Phi(m;\lambda,t)}{\Gamma(1-\lambda)} \sim C_{m,\lambda}t^{1-(m_\alpha+m_{2\alpha})}, \qquad t \to 0^+. \quad (2.38)$$

Notice that $(m_\alpha + m_{2\alpha})/(m_\alpha + 2m_{2\alpha}) = 1$ when $2\alpha \notin \Sigma$. In this case, comparing (2.36) and (2.38) with (2.35), we conclude that the estimates (2.32) near $t = 0^+$ are sharp.

When $2\alpha \in \Sigma$, then $0 \geq 1 - (m_\alpha + m_{2\alpha})/(m_\alpha + 2m_{2\alpha}) = m_\alpha/(m_\alpha + 2m_{2\alpha})$, which is ~ 0 when $m_\alpha \gg 2m_{2\alpha}$. Suppose that m is geometric and $m_{2\alpha} \neq 0$. Then $m_{2\alpha} \in 1, 3, 7$ (see [Hel78], p. 532–534). Apart from the Riemannian symmetric space associated with the exceptional group $F_{4(-20)}$ (which is the only case in which $m_{2\alpha} = 7$ occurs), m_α assumes arbitrarily big even integer numbers. Hence the condition $m_\alpha \gg 2m_{2\alpha}$ occurs geometrically. In these cases the given estimate near $t = 0^+$ is "not too bad" also in the non-reduced case.

So far we haven't discussed the estimates near $t = 0$ in the rank-one case either when $0 < m_\alpha < 2$ for Σ reduced, or when with $0 < m_{2\alpha} < 2$ for Σ non-reduced. In these cases the formula for c_0 becomes more complicated because at least one of the two square roots involved in the computations does not simplify. We only treat here the situations where $(m_\alpha, m_{2\alpha}) = (1, 0)$ and $(m_\alpha, m_{2\alpha}) = (2n, 1)$ ($n \in \mathbb{N}$). These multiplicities are geometric: the first case corresponds to Riemannian symmetric spaces locally isomorphic to the upper half-plane $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$; the second to Riemannian symmetric spaces locally isomorphic to the complex hyperbolic space $\mathrm{SU}(n+1,1)/\mathrm{SU}(n+1)$. Also for these multiplicity functions, the exact asymptotic behavior of $\Phi(m,\lambda,t)$ as $t \to 0^+$ can be read off from (2.37). When $(m_\alpha, m_{2\alpha}) = (1, 0)$, then (2.37) with $q = 0$ yields

$$\Phi((1,0);\lambda,t) \sim C_\lambda \log t, \qquad t \to 0^+.$$

Since $(c_0 + 1)/2 \in [1/2, 1)$, the estimate (2.32) cannot be sharp in this case. Suppose then $(m_\alpha, m_{2\alpha}) = (2n, 1)$ with $n \in \mathbb{N}$. Then (2.37) yields (as in the general case)

$$\Phi((2n,1);\lambda,t) \sim C_{n,\lambda}t^{-2n}, \qquad t \to 0^+.$$

We have

$$\inf_{r>1} c_{\alpha,r} = 1 - \frac{1}{n+1} \quad \text{and} \quad \inf_{r>1} c_{2\alpha,r} = -1 + \sqrt{2}.$$

Hence

$$c_0 = 1 - \frac{1}{n+1} \quad \text{and} \quad \frac{c_0+1}{2} = 1 - \frac{1}{2(n+1)}$$

for all $n \in \mathbb{N}$. Consequently, the error between exponent of the power of t in the estimate (2.32) near $t = 0^+$ and the correct asymptotic behavior of $\Phi((2n,1);\lambda,t)$ as $t \to 0^+$ is $1 - (2n+1)/(2n+2) = 1/(2n+2)$, which tends to 0 as $n \to +\infty$.

2.3 Estimates for the Θ-spherical functions

The estimates for the Harish-Chandra series established in Section 2.2 can be employed to determine estimates for the Θ-spherical functions. Before entering into the details, let us give a brief overview describing our method. We start with the formula (cf. Sections 1.4 and 1.5)

$$\varphi_\Theta(m;\lambda,a) := c_\Theta^-(m;\lambda) \sum_{w \in W_\Theta} c_\Theta^+(m;w\lambda)\Phi(m;w\lambda,a), \qquad a \in A^+, \tag{2.39}$$

which defines the Θ-spherical function $\varphi_\Theta(m;\lambda,a)$ on A^+ as a weighted sum over $w \in W_\Theta$ of the Harish-Chandra series $\Phi(m;w\lambda,a)$. For $R > 0$, let $\mathfrak{a}_{\mathbb{C}}^*(R)$ be the domain in $\mathfrak{a}_{\mathbb{C}}^*$ defined in (2.18). Let $\mathfrak{a}_{\mathbb{C},\Theta}^*(R)$ denote the largest W_Θ-invariant subset of $\mathfrak{a}_{\mathbb{C}}^*(R)$. Then every $\lambda \in \mathfrak{a}_{\mathbb{C},\Theta}^*(R)$ has the property that $w\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$ for all $w \in W_\Theta$. Hence we can apply the estimates of the previous section to all the $\Phi(m;w\lambda,a)$ occurring on the right-hand side of (2.39). Afterwards, we must estimate the various c-functions occurring as coefficients and take care of the singularities in λ. The open set $\mathfrak{a}_{\mathbb{C},\Theta}^*(R)$ intersects only finitely many singular hyperplanes of the $\Phi(m;w\lambda,a)$, of the functions $c_\Theta^\pm$ and of the numerator function n_Θ^-. The singularities in $\mathfrak{a}_{\mathbb{C},\Theta}^*(R)$ of the functions appearing in (2.39) can therefore be removed by multiplying by a suitable polynomial in λ (which will also depend on Θ and R). The estimates, initially determined only for $a \in A^+$ will then be extended by W_Θ-invariance to the entire A_Θ.

When R ranges among the positive real numbers, the domains $\mathfrak{a}_{\mathbb{C},\Theta}^*(R)$ cover $\mathfrak{a}_{\mathbb{C}}^*$. Thus our estimates will be uniform in $a \in A_\Theta$ and locally uniform in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

The most important feature of the final estimates can already be seen in this informal discussion. Indeed, for every $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $w \in W_\Theta$ the function $\delta(m;a)^{(c+1)/2}\Phi(m;w\lambda,a)$ is bounded on the walls of A^+ when $c \in [0,1)$ is chosen according to Lemma 2.2.12. Consequently, the continuous W_Θ-invariant function $\mu(m;a)^{(c+1)/2}\varphi_\Theta(m;\lambda,a)$ will be bounded on the walls of A^+, and hence bounded also on the walls of A_Θ.

For $R > 0$ and $\Theta \subset \Pi$ we define

$$\mathfrak{a}_{\mathbb{C},\Theta}^*(R) := \Big\{ \lambda \in \mathfrak{a}_{\mathbb{C}}^* : \begin{array}{ll} |\Re\langle\lambda,\alpha\rangle| < R & \text{for all } \alpha \in \langle\Theta\rangle^+, \\ \Re\langle\lambda,\alpha\rangle < R & \text{for all } \alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+ \end{array} \Big\}. \tag{2.40}$$

Figure 2.5 on p. 94 gives examples of the real parts of domains $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ for fixed $R > 0$ and variable Θ when Σ is a root system of type A_2.

Remark 2.3.1 Observe that $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ coincides with the set of all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ satisfying

$$\begin{aligned} |\Re\langle\lambda,\alpha\rangle| &< R \quad \text{for all } \alpha \in \langle\Theta\rangle^+_{\mathrm{u}}, \\ \Re\langle\lambda,\alpha\rangle &< R \quad \text{for all } \alpha \in \Sigma^+_{\mathrm{u}} \setminus \langle\Theta\rangle^+_{\mathrm{u}}. \end{aligned}$$

Therefore $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ is uniquely determined by means of the positive unmultipliable roots in Σ.

Lemma 2.3.2 *Let $\Theta \subset \Pi$ be fixed. Then for all $R > 0$ the open set $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ is the largest W_Θ-invariant subset of $\mathfrak{a}^*_{\mathbb{C}}(R)$. Moreover*

$$\mathfrak{a}^*_{\mathbb{C}} = \cup_{R>0}\mathfrak{a}^*_{\mathbb{C},\Theta}(R).$$

Proof. Let us first prove that $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ is W_Θ-invariant by showing that $r_j(\mathfrak{a}^*_{\mathbb{C},\Theta}(R)) \subset \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ for every reflection r_j corresponding to a simple root $\alpha_j \in \Theta$. Let $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$. Suppose $\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$. Then also $r_j\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$. Consequently $\Re\langle r_j\lambda,\alpha\rangle = \Re\langle\lambda, r_j\alpha\rangle < R$. If $\alpha \in \langle\Theta\rangle^+$ then $r_j\alpha \in \langle\Theta\rangle$. Hence $|\Re\langle r_j\lambda,\alpha\rangle| = |\Re\langle\lambda, r_j\alpha\rangle| < R$. In any case, we find that $r_j\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$.

To prove the maximal W_Θ-invariance, we show that

$$\mathfrak{a}^*_{\mathbb{C},\Theta}(R) = \cap_{w\in W_\Theta} w\big(\mathfrak{a}^*_{\mathbb{C}}(R)\big),$$

The right-hand side is the largest W_Θ-invariant subset of $\mathfrak{a}^*_{\mathbb{C}}(R)$. It therefore contains $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ because of the first part of the proof. Conversely, suppose λ belongs to the set on the right-hand side. In particular, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}(R)$, hence $\Re\langle\lambda,\alpha\rangle < R$ for all $\alpha \in \Sigma^+$. Suppose $\alpha \in \langle\Theta\rangle^+$. Since $\lambda \in r_\alpha\big(\mathfrak{a}^*_{\mathbb{C}}(R)\big)$, there is $\eta \in \mathfrak{a}^*_{\mathbb{C}}(R)$ with $\lambda = r_\alpha\eta$. Hence $\Re\langle\lambda,\alpha\rangle = \Re\langle\eta, r_\alpha\alpha\rangle = -\Re\langle\eta,\alpha\rangle > -R$. Thus $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$.

That the $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ cover $\mathfrak{a}^*_{\mathbb{C}}$ is clear. □

Lemma 2.3.3 *Let $R > 0$, $r \in \mathbb{R}$ and $\alpha \in \Sigma$. Suppose the set $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ intersects the complex hyperplane $\mathcal{H}_{r,\alpha} = \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} : \lambda_\alpha = r\}$.*

(a) *If $\alpha \in \langle\Theta\rangle$, then $|r| < R/\langle\alpha,\alpha\rangle$.*

(b) *If $\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$, then $r < R/\langle\alpha,\alpha\rangle$.*

Proof. (a) Recall the notation $\lambda_\alpha := \langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle$. Suppose $\alpha \in \langle\Theta\rangle$. Since $\mathcal{H}_{r,-\alpha} = \mathcal{H}_{-r,\alpha}$, by possibly replacing α with $-\alpha$, we can assume that $\alpha \in \langle\Theta\rangle^+$. In this case, if $\lambda \in \mathcal{H}_{r,\alpha} \cap \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$, then

$$|r|\,\langle\alpha,\alpha\rangle = |\Re\langle\lambda,\alpha\rangle| < R.$$

(b) If $\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$ and $\lambda \in \mathcal{H}_{r,\alpha} \cap \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$, then

$$r\langle\alpha,\alpha\rangle = \Re\langle\lambda,\alpha\rangle < R.$$

□

Corollary 2.3.4 *Suppose $R > 0$. Then the set $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ intersects only finitely many singular hyperplanes of the function $c^+_\Theta(m; w\lambda)$ for all $w \in W_\Theta$.*

Proof. Since $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ is a W_Θ-invariant domain, it is enough to prove that it intersects only finitely many singular hyperplanes of $c^+_\Theta(m; \lambda)$.

Recall from Lemma 1.4.9 that the singularities of $c^+_\Theta(m; \lambda)$ are at most simple poles located along the hyperplanes $\mathcal{H}_{-n,\alpha} = \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} : \lambda_\alpha = -n\}$ with $\alpha \in \langle\Theta\rangle^+_{\mathrm{i}}$ and $n \in \mathbb{N}_0$. Suppose $\alpha \in \langle\Theta\rangle^+_{\mathrm{i}}$ is fixed. Then Lemma 2.3.3 proves that the intersection $\mathcal{H}_{-n,\alpha} \cap \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ can be nonempty at most for $n \in \{0, 1, \ldots, R_\alpha\}$, where R_α is the largest nonnegative integer smaller than $R/\langle\alpha, \alpha\rangle$. □

Extending to all $\alpha \in \langle\Theta\rangle_{\mathrm{i}}$ the notation introduced in the proof of Corollary 2.3.4 for the elements in $\langle\Theta\rangle^+_{\mathrm{i}}$, we set for every $R > 0$ and $\alpha \in \langle\Theta\rangle_{\mathrm{i}}$

$$R_\alpha := \left[\frac{R}{\langle\alpha, \alpha\rangle}\right], \tag{2.41}$$

where for $x \in (0, +\infty)$ we indicate by $[x]$ the largest nonnegative integer smaller than x. We define

$$\pi_{\Theta,R}(\lambda) := \prod_{\alpha\in\langle\Theta\rangle_{\mathrm{i}}} \prod_{n=0}^{R_\alpha} (\lambda_\alpha + n). \tag{2.42}$$

Corollary 2.3.5 *Suppose $R > 0$ is fixed. Then $\pi_{\Theta,R}(\lambda)$ is a W_Θ-invariant polynomial function in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ with the property that*

$$\pi_{\Theta,R}(\lambda)c^+_\Theta(m; w\lambda)$$

is holomorphic in $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ for all $w \in W_\Theta$.

Proof. The W_Θ-invariance is an immediate consequence of the facts that W_Θ maps $\langle\Theta\rangle_{\mathrm{i}}$ onto itself and that $R_{w\alpha} = [R/\langle w\alpha, w\alpha\rangle] = [R/\langle\alpha, \alpha\rangle] = R_\alpha$ for all $\alpha \in \langle\Theta\rangle_{\mathrm{i}}$.

Because of the defining formula (1.37), to prove the statement on the holomorphy of the function $\pi_{\Theta,R}(\lambda)c^+_\Theta(m; w\lambda)$ in $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$, it suffices to check that the function

$$\pi_{\Theta,R}(\lambda)\Gamma\big((w\lambda)_\alpha\big) = \pi_{\Theta,R}(w\lambda)\Gamma\big((w\lambda)_\alpha\big)$$

is holomorphic in $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ for all $\alpha \in \langle\Theta\rangle^+_{\mathrm{i}}$. Observe that

$$\Big[\prod_{n=0}^{R_\alpha} \big((w\lambda)_\alpha + n\big)\Big]\Gamma\big((w\lambda)_\alpha\big) = \Gamma\big((w\lambda)_\alpha + R_\alpha + 1\big) = \Gamma(\lambda_{w^{-1}\alpha} + R_\alpha + 1)$$

has singularities along the hyperplanes $\mathcal{H}_{-(R_\alpha+n),w^{-1}\alpha}$ with $n \in \mathbb{N}$. Since $w^{-1}\alpha \in \langle\Theta\rangle$ and $|-(R_\alpha + n)| \geq R_\alpha + 1 \geq R/\langle\alpha, \alpha\rangle$, these hyperplanes do not intersect $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ by Lemma 2.3.3. □

Estimates for the functions $\pi_{\Theta,R}(\lambda)c_\Theta^+(m;w\lambda)$ on $\mathfrak{a}_{\mathbb{C},\Theta}^*(R)$ are obtained from the classical asymptotics of ratios of gamma functions: For arbitrarily fixed $a, b \in \mathbb{C}$ and any small $\varepsilon > 0$ we have

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left[1 + O(z^{-1})\right] \tag{2.43}$$

for all $z \in \mathbb{C}$ with $|\arg z| \le \pi - \varepsilon < \pi$. The constant implied by the symbol O depends on both a and b (see e.g. [E$^+$53], 1.18(4)).

Lemma 2.3.6 *Let $R > 0$ be fixed. Then there are positive constants $C_{\Theta,R,m}$ and $r(\Theta,R,m)$ such that*

$$\left|\pi_{\Theta,R}(\lambda)c_\Theta^+(m;w\lambda)\right| \le C_{\Theta,R,m}(1+|\lambda|)^{r(\Theta,R,m)}$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C},\Theta}^*(R)$ and all $w \in W_\Theta$.

Proof. Set

$$\pi_{\Theta,R}^-(\lambda) := \prod_{\alpha\in-\langle\Theta\rangle_1^+}\prod_{n=0}^{R_\alpha}(\lambda_\alpha + n).$$

Then by (1.37)

$$\pi_{\Theta,R}(\lambda)c_\Theta^+(m;\lambda) = \pi_{\Theta,R}^-(\lambda)\prod_{\alpha\in\langle\Theta\rangle_1^+}\sqrt{\pi}\,\frac{2^{-(\lambda_\alpha+m_\alpha/2)+1}\,\Gamma(\lambda_\alpha+R_\alpha+1)}{\Gamma\left(\frac{\lambda_\alpha}{2}+\frac{m_\alpha}{4}+\frac12\right)\Gamma\left(\frac{\lambda_\alpha}{2}+\frac{m_\alpha}{4}+\frac{m_{2\alpha}}{2}\right)}.$$

Because of the duplication formula (1.34) for the gamma function,

$$\Gamma(\lambda_\alpha+R_\alpha+1) = \frac{2^{\lambda_\alpha+R_\alpha}}{\sqrt{\pi}}\,\Gamma\Big(\frac{\lambda_\alpha+R_\alpha+1}{2}\Big)\Gamma\Big(\frac{\lambda_\alpha+R_\alpha}{2}+1\Big).$$

Hence (2.43) yields

$$\begin{aligned}
&\left|\sqrt{\pi}\,\frac{2^{-(\lambda_\alpha+m_\alpha/2)+1}\,\Gamma(\lambda_\alpha+R_\alpha+1)}{\Gamma\left(\frac{\lambda_\alpha}{2}+\frac{m_\alpha}{4}+\frac12\right)\Gamma\left(\frac{\lambda_\alpha}{2}+\frac{m_\alpha}{4}+\frac{m_{2\alpha}}{2}\right)}\right| \\
&= 2^{R_\alpha-m_\alpha/2+1}\left|\frac{\Gamma\left(\frac{\lambda_\alpha}{2}+\frac{R_\alpha}{2}+1-\frac12\right)}{\Gamma\left(\frac{\lambda_\alpha}{2}+\frac{R_\alpha}{2}+1+\left(\frac{m_\alpha}{4}-\frac{R_\alpha}{2}-\frac12\right)\right)}\right|\left|\frac{\Gamma\left(\frac{\lambda_\alpha}{2}+\frac{R_\alpha}{2}+1\right)}{\Gamma\left(\frac{\lambda_\alpha}{2}+\frac{R_\alpha}{2}+1+\left(\frac{m_\alpha}{4}+\frac{m_{2\alpha}}{2}-\frac{R_\alpha}{2}-1\right)\right)}\right| \\
&= 2^{R_\alpha-m_\alpha/2+1}\left|\frac{\lambda_\alpha}{2}+\frac{R_\alpha}{2}+1\right|^{R_\alpha+1-(m_\alpha+m_{2\alpha})/2}\left[1+O\Big(\big(\frac{\lambda_\alpha}{2}+\frac{R_\alpha}{2}+1\big)^{-1}\Big)\right] \\
&= 2^{m_{2\alpha}/2}\,|\lambda_\alpha+R_\alpha+2|^{R_\alpha+1-(m_\alpha+m_{2\alpha})/2}\left[1+O\Big(\big(\frac{\lambda_\alpha}{2}+\frac{R_\alpha}{2}+1\big)^{-1}\Big)\right]
\end{aligned} \tag{2.44}$$

for all λ with $\left|\arg\left(\frac{\lambda_\alpha}{2}+\frac{R_\alpha}{2}+1\right)\right| \le \pi-\varepsilon<\pi$.

If $\lambda \in \mathfrak{a}_{\mathbb{C},\Theta}^*(R)$, then

$$\Re(\lambda_\alpha+R_\alpha+2) = \Re\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}+R_\alpha+2 > -\frac{R}{\langle\alpha,\alpha\rangle}+\left[\frac{R}{\langle\alpha,\alpha\rangle}\right]+2 \ge 1,$$

since $0 < x - [x] \leq 1$ for all $x \in (0, +\infty)$. This has two consequences. First, that

$$\left|\arg\left(\frac{\lambda_\alpha}{2} + \frac{R_\alpha}{2} + 1\right)\right| < \frac{\pi}{2}$$

for all $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$. Hence the estimate (2.44) holds for all $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$. Second, that

$$|\lambda_\alpha + R_\alpha + 2| \geq 1$$

for all $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$. Hence there is a constant $C_{\alpha,R,m} > 0$ (depending on α, R, m but not on λ) such that

$$\left|1 + O\Big(\big(\frac{\lambda_\alpha}{2} + \frac{R_\alpha}{2} + 1\big)^{-1}\Big)\right| \leq C_{\alpha,R,m}$$

for all $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$.

Suppose $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ is arbitrarily fixed. If $S := R_\alpha + 1 - (m_\alpha + m_{2\alpha})/2 \leq 0$, then $|\lambda_\alpha + R_\alpha + 2|^S \leq 1$. Suppose then $S > 0$. Notice that there is a constant $C_{\alpha,R}$ (independent of λ) such that

$$|\lambda_\alpha + R_\alpha + 2| \leq \frac{|\lambda|}{|\alpha|} + R_\alpha + 2 \leq C_{\alpha,R}(1 + |\lambda|).$$

Hence

$$|\lambda_\alpha + R_\alpha + 2|^S \leq C^S_{\alpha,R}(1 + |\lambda|)^S.$$

Taking all of this into account, we conclude therefore that there exist positive constants $C_{\Theta,R,m}$ and $r(\Theta, R, m)$ for which

$$\left|\pi_{\Theta,R}(\lambda)c^+_\Theta(m;\lambda)\right| \leq C_{\Theta,R,m}(1 + |\lambda|)^{r(\Theta,R,m)}$$

for all $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$. The constant $r(\Theta, R, m)$ can be taken equal to

$$\sum_{\alpha \in -\langle\Theta\rangle^+_{\mathrm{i}}} (R_\alpha + 1) + {\sum_{\alpha \in \langle\Theta\rangle^+_{\mathrm{i}}}}' \left(R_\alpha + 1 - \frac{m_\alpha + m_{2\alpha}}{2}\right),$$

the second sum being over all α's with $R_\alpha + 1 - \frac{m_\alpha + m_{2\alpha}}{2} > 0$.

This proves the required inequality for $w = \mathrm{id} \in W_\Theta$. Observe that $|w\lambda| = |\lambda|$ for all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $w \in W$. If $w \in W_\Theta$ is arbitrary, we therefore obtain by the W_Θ-invariance of $\pi_{\Theta,R}$ and of $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$

$$\begin{aligned}\left|\pi_{\Theta,R}(\lambda)c^+_\Theta(m;\lambda)\right| &= \left|\pi_{\Theta,R}(w\lambda)c^+_\Theta(m;w\lambda)\right| \\ &\leq C_{\Theta,R,m}(1 + |w\lambda|)^{r(\Theta,R,m)} \\ &= C_{\Theta,R,m}(1 + |\lambda|)^{r(\Theta,R,m)}\end{aligned}$$

for all $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$. □

Recall the notation $c^-_\Theta(m;\lambda) = n^-_\Theta(m;\lambda)/d^-_\Theta(m;\lambda)$ from Definition 1.4.6. Both functions $n^-_\Theta(m;\lambda)$ and $d^-_\Theta(m;\lambda)$ are W_Θ-invariant. Let $R > 0$ be fixed. By definition, the

singularities of c_Θ^- in $\mathfrak{a}_{\mathbb{C},\Theta}^*(R)$ are (counted with multiplicities) among those of n_Θ^-. By Theorem 1.5.11, also the singularities of the Θ-spherical functions $\varphi_\Theta(m;\lambda,a)$ are (counted with multiplicities) among those of n_Θ^-. This holds for all a in a W_Θ-invariant tubular neighborhood of A_Θ in $A_{\mathbb{C}}$. However, we are now restricting ourselves only to $a \in A_\Theta$.

According to Lemma 1.4.9 the singularities of n_Θ^- are poles located along the hyperplanes

$$\begin{aligned} &\mathcal{H}_{-\frac{m_\alpha}{2}+(2n-1),\alpha} && \text{with } \alpha \in \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+ \text{ and } n \in \mathbb{N}, \\ &\mathcal{H}_{-\frac{m_\alpha}{2}-m_{2\alpha}+2n,\alpha} && \text{with } \alpha \in \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+ \text{ and } n \in \mathbb{N}. \end{aligned}$$

Each listed hyperplane contributes a simple pole to n_Θ^-. Nevertheless, when Σ is not reduced, the resulting family of singularities might not consist only of simple poles, since some pairs of the listed hyperplanes can coincide. The following corollary is an immediate consequence of the above list and Lemma 2.3.3.

Corollary 2.3.7 *Suppose $R > 0$. Then $\mathfrak{a}_{\mathbb{C},\Theta}^*(R)$ intersects only finitely many singular hyperplanes of the numerator function $n_\Theta^-(m;\lambda)$. More precisely, let $\alpha \in \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+$. Then $\mathfrak{a}_{\mathbb{C},\Theta}^*(R)$ intersects:*

(a) *no hyperplanes $\mathcal{H}_{-\frac{m_\alpha}{2}+(2n-1),\alpha}$ ($n \in \mathbb{N}$), if $R \le \langle\alpha,\alpha\rangle(1 - m_\alpha/2)$ (which is possible only when $m_\alpha < 2$);*

(b) *at most the hyperplanes $\mathcal{H}_{-\frac{m_\alpha}{2}+(2n-1),\alpha}$ with $n = 1, 2, \ldots, R_{\alpha,1}$, if $R > \langle\alpha,\alpha\rangle(1 - m_\alpha/2)$, where $R_{\alpha,1} := \left[\frac{1}{2}\left(\frac{m_\alpha}{2} + \frac{R}{\langle\alpha,\alpha\rangle} + 1\right)\right]$;*

(c) *no hyperplanes $\mathcal{H}_{-\frac{m_\alpha}{2}-m_{2\alpha}+2n,\alpha}$ ($n \in \mathbb{N}$), if $R \le \langle\alpha,\alpha\rangle(2 - m_\alpha/2 - m_{2\alpha})$ (which is possible only when $m_\alpha/2 + m_{2\alpha} < 2$);*

(d) *at most the hyperplanes $\mathcal{H}_{-\frac{m_\alpha}{2}-m_{2\alpha}+2n,\alpha}$ with $n = 1, 2, \ldots, R_{\alpha,2}$, if $R > \langle\alpha,\alpha\rangle(2 - m_\alpha/2 - m_{2\alpha})$, where $R_{\alpha,2} := \left[\frac{1}{2}\left(\frac{m_\alpha}{2} + m_{2\alpha} + \frac{R}{\langle\alpha,\alpha\rangle}\right)\right]$.*

□

We set $R_{\alpha,1} := 0$ in case (a) of Corollary 2.3.7, and $R_{\alpha,2} := 0$ in case (c). With the given $R_{\alpha,1}$ and $R_{\alpha,2}$ for the cases (b) and (d), we define

$$n_{\Theta,R}^-(m;\lambda) := \prod_{\alpha\in\Sigma_{\mathrm{i}}^+\setminus\langle\Theta\rangle_{\mathrm{i}}^+} \prod_{n_1=1}^{R_{\alpha,1}} \left(-\frac{\lambda_\alpha}{2} - \frac{m_\alpha}{4} - \frac{1}{2} + n_1\right) \prod_{n_2=1}^{R_{\alpha,2}} \left(-\frac{\lambda_\alpha}{2} - \frac{m_\alpha}{4} - \frac{m_{2\alpha}}{2} + n_2\right), \tag{2.45}$$

with the usual convention that empty products are equal to 1.

Corollary 2.3.8 *Suppose $R > 0$ is fixed. Then $n_{\Theta,R}^-(m;\lambda)$ is a W_Θ-invariant polynomial function in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with the property that*

$$n_{\Theta,R}^-(m;\lambda)\varphi_\Theta(m;\lambda,a)$$

is holomorphic in $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ for all $a \in A_\Theta$.

In fact, the same property holds for all a in the W_Θ-invariant tubular neighborhood U_Θ of A_Θ in $A_{\mathbb{C}}$ of Theorem 1.5.11.

Proof. Because of Lemma 1.1.2, the group W_Θ maps $\Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+$ onto itself. The W-invariance of the inner product and of the multiplicity function m then ensure that $R_{\alpha,1}$ and $R_{\alpha,2}$ are W-invariant. Thus $n^-_{\Theta,R}(m;\lambda)$ is W_Θ-invariant.

To prove that $n^-_{\Theta,R}(m;\lambda)\varphi_\Theta(m;\lambda,a)$ is holomorphic in $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$, it is enough to show that so is the function $n^-_{\Theta,R}(m;\lambda)n^-_\Theta(m;\lambda)$. Using (1.45) and (2.45) together with the functional relation $z\Gamma(z) = \Gamma(z+1)$, one easily verifies the equality

$$\begin{aligned} &n^-_{\Theta,R}(m;\lambda)n^-_\Theta(m;\lambda) \\ &= \prod_{\alpha\in\Sigma_{\mathrm{i}}^+\setminus\langle\Theta\rangle_{\mathrm{i}}^+} \Gamma\left(-\frac{\lambda_\alpha}{2} - \frac{m_\alpha}{4} + \frac{1}{2} + R_{\alpha,1}\right)\Gamma\left(-\frac{\lambda_\alpha}{2} - \frac{m_\alpha}{4} - \frac{m_{2\alpha}}{2} + 1 + R_{\alpha,2}\right). \quad (2.46) \end{aligned}$$

The conditions defining $R_{\alpha,1}$ and $R_{\alpha,2}$ ensure that whenever $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ satisfies $\Re\langle\lambda,\alpha\rangle < R$, one has

$$\begin{aligned} -\Re\frac{\lambda_\alpha}{2} - \frac{m_\alpha}{4} + \frac{1}{2} + R_{\alpha,1} > 0, \\ -\Re\frac{\lambda_\alpha}{2} - \frac{m_\alpha}{4} - \frac{m_{2\alpha}}{2} + 1 + R_{\alpha,2} > 0. \end{aligned}$$

The product (2.46) is therefore nonsingular in $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$. □

Lemma 2.3.9 *Let $R > 0$ be fixed. There there are positive constants $C'_{\Theta,R,m}$ and $r'(\Theta,R,m)$ such that*

$$\left|n^-_{\Theta,R}(m;\lambda)c^-_\Theta(m;\lambda)\right| \le C'_{\Theta,R,m}(1+|\lambda|)^{r'(\Theta,R,m)}$$

for all $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$.

Proof. The argument we need is similar to the one used in the proof of Lemma 2.3.6. We therefore only give a brief outline.

Observe first that

$$n^-_{\Theta,R}(m;\lambda)c^-_\Theta(m;\lambda) = \frac{n^-_{\Theta,R}(m;\lambda)n^-_\Theta(m;\lambda)}{d^-_\Theta(m;\lambda)},$$

in which the numerator is given by (2.46). Formula (1.46) for the denominator d^-_Θ can be rewritten using the duplication formula (1.34) for the gamma function. Indeed,

$$\sqrt{\pi}\,2^{\lambda_\alpha+m_\alpha/2}\,\Gamma(-\lambda_\alpha+1) = 2^{m_\alpha/2}\,\Gamma\Big(\frac{-\lambda_\alpha+1}{2}\Big)\Gamma\Big(-\frac{\lambda_\alpha}{2}+1\Big).$$

The asymptotics (2.43) yield

$$\begin{aligned}
&\left|\frac{\Gamma\Big(-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}+\frac{1}{2}+R_{\alpha,1}\Big)\Gamma\Big(-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}-\frac{m_{2\alpha}}{2}+1+R_{\alpha,2}\Big)}{2^{m_\alpha/2}\,\Gamma\Big(\frac{-\lambda_\alpha+1}{2}\Big)\Gamma\Big(\frac{-\lambda_\alpha}{2}+1\Big)}\right| \\
&= 2^{-m_\alpha/2}\left|-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}+\frac{3}{2}+R_{\alpha,1}\right|^{R_{\alpha,1}-m_\alpha/4}\left[1+O\Big(\big(-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}+\frac{3}{2}+R_{\alpha,1}\big)^{-1}\Big)\right] \\
&\times\left|-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}-\frac{m_{2\alpha}}{2}+\frac{3}{2}+R_{\alpha,2}\right|^{R_{\alpha,2}-m_\alpha/4-m_{2\alpha}/2}\left[1+O\Big(\big(-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}-\frac{m_{2\alpha}}{2}+\frac{3}{2}+R_{\alpha,2}\big)^{-1}\Big)\right] \\
&= 2^{m_{2\alpha}/2-(R_{\alpha,1}+R_{\alpha,2})}\left|-\lambda_\alpha-\frac{m_\alpha}{2}+3+2R_{\alpha,1}\right|^{R_{\alpha,1}-m_\alpha/4}\left[1+O\Big(\big(-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}+\frac{3}{2}+R_{\alpha,1}\big)^{-1}\Big)\right] \\
&\times\left|-\lambda_\alpha-\frac{m_\alpha}{2}-m_{2\alpha}+3+2R_{\alpha,2}\right|^{R_{\alpha,2}-m_\alpha/4-m_{2\alpha}/2}\left[1+O\Big(\big(-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}-\frac{m_{2\alpha}}{2}+\frac{3}{2}+R_{\alpha,2}\big)^{-1}\Big)\right]
\end{aligned} \tag{2.47}$$

uniformly for all λ satisfying for some $\varepsilon > 0$

$$\begin{aligned}
\left|-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}+\frac{3}{2}+R_{\alpha,1}\right| &\le \pi-\varepsilon<\pi, \\
\left|-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}-\frac{m_{2\alpha}}{2}+\frac{3}{2}+R_{\alpha,2}\right| &\le \pi-\varepsilon<\pi.
\end{aligned}$$

One can easily check from the definition of $R_{\alpha,1}$ and $R_{\alpha,2}$ that for $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$

$$\begin{aligned}
\Re\left(-\lambda_\alpha-\frac{m_\alpha}{2}+3+2R_{\alpha,1}\right) &\ge 1, \\
\Re\left(-\lambda_\alpha-\frac{m_\alpha}{2}-m_{2\alpha}+3+2R_{\alpha,2}\right) &\ge 1.
\end{aligned}$$

since $0 < 2(x-[x]) \le 2$ for all $x \in (0,+\infty)$. Again as in the proof of Lemma 2.3.6, this has two consequences. First, that the estimate (2.47) holds for all $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$. Second, that

$$\begin{aligned}
\left|-\lambda_\alpha-\frac{m_\alpha}{2}+3+2R_{\alpha,1}\right| &\ge 1, \\
\left|-\lambda_\alpha-\frac{m_\alpha}{2}-m_{2\alpha}+3+2R_{\alpha,2}\right| &\ge 1
\end{aligned}$$

for all $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$. Hence there is a constant $C'_{\alpha,R,m} > 0$ (depending on α, R, m but not on λ) such that

$$\begin{aligned}
\left|1+O\Big(\big(-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}+\frac{3}{2}+R_{\alpha,1}\big)^{-1}\Big)\right| &\le C'_{\alpha,R,m} \\
\left|1+O\Big(\big(-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}-\frac{m_{2\alpha}}{2}+\frac{3}{2}+R_{\alpha,2}\big)^{-1}\Big)\right| &\le C'_{\alpha,R,m}.
\end{aligned}$$

for all $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$. The claim then follows as in Lemma 2.3.6. □

Theorem 2.2.18 proves estimates for the Harish-Chandra series $\Phi(m;\lambda,a)$ for all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}(R)$ and $a \in A^+$: multiplication by a certain polynomial $p_R(\lambda)$ cancels all λ-singularities

of $\Phi(m;\lambda,a)$ in $\mathfrak{a}_{\mathbb{C}}^*(R)$; multiplication by $\delta(m;a)^{(c+1)/2}$, for a suitable $c \in [0,1)$, cancels the singularities along the walls of A^+. According to (2.20), the polynomial p_R is given by

$$p_R(\lambda) := \prod_{\eta \in \mathcal{X}_R} \langle \eta - 2\lambda, \eta \rangle,$$

where $\mathcal{X}_R$ is the finite set of the $\eta \in 2\Lambda \setminus \{0\}$ for which $\mathcal{H}_\eta \cap \mathfrak{a}_{\mathbb{C}}^*(2R) \neq 0$. The set $\mathcal{X}_R$ is empty when $R > 0$ is sufficiently small, for instance when $R \leq 1/(2Q^2)$ with Q given by (2.17). We make $p_R(\lambda)$ invariant under W_Θ by defining

$$p_{\Theta,R}(\lambda) := \prod_{w \in W_\Theta} p_R(w\lambda). \tag{2.48}$$

Then $p_{\Theta,R}(\lambda)$ is a polynomial in λ of degree $\deg p_{\Theta,R} = |W_\Theta| \deg p_R$. Moreover, $p_{\Theta,R} \equiv 1$ whenever so is p_R.

By restricting the variable λ to the W_Θ-invariant domain $\mathfrak{a}_{\mathbb{C},\Theta}^*(R) \subset \mathfrak{a}_{\mathbb{C}}^*(R)$, we can apply Theorem 2.2.18 and obtain estimates for $\Phi(m; w\lambda, a)$ for all $w \in W_\Theta$.

Corollary 2.3.10 *Let $R > 0$ and let $p_{\Theta,R}$ be the polynomial defined in (2.48). Suppose $m \in \mathcal{M}^+$ and let $\delta(m;a)$ be defined by (1.22). Then there exists $c \in [0,1)$ (which depends only on the multiplicity function m and which can be computed from Lemma 2.2.12) and a constant $C_{R,c,m} > 0$ such that*

$$\left| p_{\Theta,R}(\lambda) \delta(m;a)^{(c+1)/2} \Phi(m; w\lambda, a) \right| \leq C_{R,c,m} (1 + |\lambda|)^{\deg p_{\Theta,R}} e^{(c\rho(m) + \Re w\lambda)(\log a)} \tag{2.49}$$

for all $a \in A^+$ and $\lambda \in \mathfrak{a}_{\mathbb{C},\Theta}^*(R)$.

We can choose $p_{\Theta,R} \equiv 1$ for all sufficiently small $R > 0$ (e.g. $R \leq 1/(2Q^2)$ with Q as in (2.17)). □

The right-hand side of (2.49) can be made W_Θ-invariant by replacing $\Re w\lambda(\log a)$ with

$$\Re W_\Theta \lambda(\log a) := \max_{w \in W_\Theta} \Re w\lambda(\log a) \tag{2.50}$$

and $a \in A^+$ with the element a_Θ of the W_Θ-orbit of a lying in $\overline{A^+}$, that is

$$\{a_\Theta\} = W_\Theta a \cap \overline{A^+}. \tag{2.51}$$

Notice that $\Re W_\Theta \lambda(\log a) = \Re W_\Theta \lambda(\log a_\Theta)$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $\alpha \in A_\Theta$.

To keep the W_Θ-invariance when dealing with the Θ-spherical functions, we must moreover replace the function $\delta(m)$ with the W_Θ-invariant continuous function $\mu(m)$, which agrees with $\delta(m)$ on A^+. See (2.1).

Combining all estimates proven in this section, we reach the following estimate for the Θ-spherical functions.

Theorem 2.3.11 *Suppose $m \in \mathcal{M}^+$ and $R > 0$. Let $\pi_{\Theta,R}(\lambda)$, $p_{\Theta,R}(\lambda)$, $n_{\Theta,R}^-(m;\lambda)$ be the polynomials defined in (2.42), (2.48) and (2.45), respectively. Let $\mu(m;a)$ be defined by*

(2.1). Finally, let $c \in [0,1)$ be the constant (depending only on the multiplicity function m) which can be computed from Lemma 2.2.12. Then there are constants $C_{\Theta,R,c,m} > 0$ and $d(\Theta,R,m) \geq 0$ such that

$$\begin{aligned}\left|\pi_{\Theta,R}(\lambda)p_{\Theta,R}(\lambda)n^-_{\Theta,R}(m;\lambda)\mu(m;a)^{(c+1)/2}\varphi_\Theta(m;\lambda,a)\right| \\ \leq C_{\Theta,R,c,m}(1+|\lambda|)^{d(\Theta,R,m)}e^{(c\rho(m)+\Re W_\Theta\lambda)(\log a_\Theta)}\end{aligned} \tag{2.52}$$

for all $a \in A_\Theta$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$.

We can choose $p_{\Theta,R} \equiv 1$ for all sufficiently small $R > 0$ (e.g. $R \leq 1/(2Q^2)$ with Q as in (2.17)). Moreover, we have $n^-_{\Theta,R} \equiv 1$ for all R when $\Theta = \Pi$, and $\pi_{\Theta,R} \equiv 1$ for all R when $\Theta = \emptyset$.

Proof. For all $a \in A^+$ and for generic $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ the Θ-spherical function φ_Θ is defined by (1.48), which implies the equality

$$\begin{aligned}&\pi_{\Theta,R}(\lambda)p_{\Theta,R}(\lambda)n^-_{\Theta,R}(m;\lambda)\mu(m;a)^{(c+1)/2}\varphi_\Theta(m;\lambda,a) \\ &= \left[n^-_{\Theta,R}(m;\lambda)c^-_\Theta(m;\lambda)\right]\sum_{w\in W_\Theta}\left[\pi_{\Theta,R}(\lambda)c^+_\Theta(m;w\lambda)\right]\left[p_{\Theta,R}(\lambda)\delta(m;a)^{(c+1)/2}\Phi(m;w\lambda,a)\right]\end{aligned}$$

Lemmas 2.3.9 and 2.3.6 together with Corollary 2.3.10 prove the required estimate (2.52) for generic $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ and $a \in A^+$. The left-hand side of (2.52) is a holomorphic function of $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ by Corollary 2.3.8. Moreover, it is continuous and W_Θ-invariant in $a \in A_\Theta$. The estimate therefore extends by continuity for all $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ and $a \in A_\Theta$. □

The polynomials $\pi_{\Theta,R}(\lambda)$and $p_{\Theta,R}(\lambda)$ appearing on the left-hand side of (2.52) are in fact unnecessary. They would allow $n^-_{\Theta,R}(m;\lambda)\varphi_\Theta(m;\lambda,a)$ to have singularities in $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$, but we know from Corollary 2.3.8 that this function is holomorphic in $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ for all $a \in A_\Theta$. It is possible to apply Corollary 2.2.2 and improve the estimates in λ by removing these superfluous polynomials, but in doing so we somehow worsen the estimates in the a-variable for large a.

Corollary 2.3.12 *Keep the notation of Theorem 2.3.11. Then there are constants $\varepsilon_R > 0$ and $d'(\Theta,R,m) \geq 0$ with the following property: For every $\varepsilon \in (0,\varepsilon_R]$ there is a constant $C_{\Theta,R,c,m,\varepsilon} > 0$ such that*

$$\begin{aligned}\left|n^-_{\Theta,R}(m;\lambda)\mu(m;a)^{(c+1)/2}\varphi_\Theta(m;\lambda,a)\right| \\ \leq C_{\Theta,R,c,m,\varepsilon}(1+|\lambda|)^{d'(\Theta,R,m)}e^{(c\rho(m)+\Re W_\Theta\lambda)(\log a_\Theta)+\varepsilon|\log a|}\end{aligned} \tag{2.53}$$

for all $a \in A_\Theta$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$.

Proof. Let $\varepsilon_R > 0$ be chosen so that the ε_R-neighborhood $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)_{\varepsilon_R}$ of $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ is contained in $\mathfrak{a}^*_{\mathbb{C},\Theta}(2R)$. Then $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)_\varepsilon \subset \mathfrak{a}^*_{\mathbb{C},\Theta}(2R)$ for all $\varepsilon \in (0,\varepsilon_R]$. The estimate of Theorem 2.3.11 in the larger domain $\mathfrak{a}^*_{\mathbb{C},\Theta}(2R)$ gives for all $a \in A_\Theta$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(2R)$

$$\begin{aligned}\left|q_{\Theta,R}(\lambda)n^-_{\Theta,2R}(m;\lambda)\mu(m;a)^{(c+1)/2}\varphi_\Theta(m;\lambda,a)\right| \\ \leq C_{\Theta,2R,c,m}(1+|\lambda|)^{d(\Theta,2R,m)}e^{(c\rho(m)+\Re W_\Theta\lambda)(\log a_\Theta)}\end{aligned}$$

where

$$q_{\Theta,R}(\lambda) := \pi_{\Theta,2R}(\lambda)p_{\Theta,2R}(\lambda)\frac{n^-_{\Theta,2R}(m;\lambda)}{n^-_{\Theta,R}(m;\lambda)}$$

is a polynomial. For fixed $a \in A_\Theta$, consider the elements $w \log a_\Theta$ for $w \in W_\Theta$. They satisfy $|w \log a_\Theta| = |\log a|$ for all w. The required inequality is obtained by applying Corollary 2.2.2 to the holomorphic function $H(\lambda) := q_{\Theta,R}(\lambda)n^-_{\Theta,2R}(m;\lambda)\mu(m;a)^{(c+1)/2}\varphi_\Theta(m;\lambda,a)$, the polynomial $q_{\Theta,R}(\lambda)$ and the elements $w \log a_\Theta$. □

Remark 2.3.13

(a) When $\Theta \neq \emptyset$, the estimates of Theorem 2.3.11 and Corollary 2.3.12 are not sharp in many senses. In fact, they are even getting worse when the Θ-spherical functions are becoming more regular. This can be observed in three points. First, the domain $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ for the variable λ on which the estimates hold is getting smaller when Θ increases. In particular, it is always a compact subset of $\mathfrak{a}^*_{\mathbb{C}}$ in the most regular case $\Theta = \Pi$. Second, the term $\mu(m;a)^{(c+1)/2}$ is independent of Θ. It is necessary for the case $\Theta = \emptyset$, but completely superfluous in the case $\Theta = \Pi$. The Θ-spherical function φ_Θ is regular on A_Θ. This suggests that the correct factor should be

$$\mu_\Theta(m) := \prod_{\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+} \left|e^\alpha - e^{-\alpha}\right|^{m_\alpha}. \tag{2.54}$$

Third, the polynomial factor in λ on the right-hand side of the estimates is unnecessary for $\Theta = \Pi$ (see below). However, being of polynomial nature, this additional factor gives no problems in most of the applications.

Nevertheless the estimates of Theorem 2.3.11 and Corollary 2.3.12 are the first piece of information about the behavior of the Θ-spherical functions for all values of the spectral parameter along the walls of A_Θ, also for the geometric case of spherical functions on NCC spaces. The behavior predicted along the walls will be enough to apply the Θ-spherical Fourier transform to functions without compact support in A_Θ.

(b) In Proposition 6.1(1) and Theorem 3.15 of [Opd95], Opdam has proven estimates for the hypergeometric functions associated with root systems. We state them for the functions φ_Π, which differ from the hypergeometric functions for root system only by a constant factor:

Suppose $m \in \mathcal{M}^+$. *Then*

$$|\varphi_\Pi(m;\lambda,a)| \leq |W|^{1/2}\, e^{\max_{w\in W} \Re(w\lambda(\log a))} \tag{2.55}$$

for all $a \in A$.

Opdam could even extend the estimates (in a slightly more complicated version) to a tubular neighborhood of A in $A_\mathbb{C}$. See [Opd95], Proposition 6.1(2).

The proving techniques of Opdam's estimates are very interesting: he deduces the results for the hypergeometric functions from the estimates for another family of eigenfunctions of the Dunkl-Cherednik operators, which he calls the *non-symmetric hypergeometric functions*. The Dunkl-Cherednik operators are differential-reflection operators of *first* order, and this allows Opdam to apply a clever trick to get the required estimates in the non-symmetric case. The hypergeometric functions are obtained from the non-symmetric hypergeometric functions by averaging over W, and this easily yields the estimates reported in (2.55). We refer to Section 3 in [Opd95] and Section 7 of [Opd00] for further details.

The geometric interpretation of Opdam's non-symmetric hypergeometric functions in the Riemannian case is an interesting open problem. The construction of the non-symmetric hypergeometric functions in the context of Θ-spherical functions is a challenging project with interests also for the geometric case of NCC symmetric spaces.

(c) In the geometric context of Riemannian symmetric spaces there are very sharp estimates for the spherical functions. They are known as the *Harish-Chandra estimates*. In their sharpest form, they employ *both* the integral formulas and the system of differential equations. The integral formulas provide shift inequalities which reduce to the estimation of the spherical functions for the spectral parameter $\lambda = 0$. Indeed for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $a \in A$

$$|\varphi_{\mathrm{RS}}(m;\lambda,a)| \leq \varphi_{\mathrm{RS}}(m;\Re\lambda,a) \leq e^{\max_{w\in W} \Re w\lambda(\log a)}\varphi_{\mathrm{RS}}(m;0,a) \tag{2.56}$$

(see [GV88], Proposition 4.6.1). Then, by representing $\varphi_{\mathrm{RS}}(m;0)$ in terms of the HC-series, one can prove that there is a constant $C > 0$ such that for all $a \in \overline{A^+}$

$$\varphi_{\mathrm{RS}}(m;0,a) \leq Ce^{-\rho(\log a)}(1 + \|\log a\|)^d, \tag{2.57}$$

where d is the number of indivisible roots in Σ (that is, the number of reflecting hyperplanes in W) and $\|\cdot\|$ is any W-invariant norm on $\mathfrak{a}_{\mathbb{C}}$ (cf. *loc. cit.*, Theorem 4.6.4). The integral exponent d is optimal (cf. *loc. cit.*, Theorem 4.6.6). The estimate (2.57) is the fundamental estimate of the harmonic analysis on the noncompact semisimple Lie group G.

(d) The estimation techniques of the Riemannian case cannot be applied in the context of hypergeometric functions associated with root systems (and therefore in the general context of Θ-spherical functions) because of the absence of integral formulas.

They cannot be applied even in the geometric setting of the NCC spaces, where the integral formulas are not sufficiently explicit. For arbitrary values of the spectral parameter λ, these integral formulas have been recently obtained in [ÓP01a] by the method of meromorphic continuation via Bernstein-Sato polynomials. They have been applied in [AÓS00] to get some estimates for the spherical functions. However, due to

the insufficient information on the Bernstein-Sato polynomials and their corresponding differential operators, the estimates in [AÓS00] could not predict any behavior of the spherical functions along the walls of A_Θ.

We now apply the estimates of Theorem 2.3.11 and Corollary 2.3.12 and discuss the applicability of the Θ-spherical Fourier transform to some classes of W_Θ-invariant functions on A_Θ. Recall from Definitions 2.1.1 and 2.1.2 that for $\Theta \subset \Pi$ and for a sufficiently regular W_Θ-invariant function f on A_Θ, the Θ-spherical Fourier transform and the normalized Θ-spherical Fourier transform of f associated with the data $(\mathfrak{a}, \Sigma, m)$ are the W_Θ-invariant functions $\mathcal{F}_\Theta f(m)$ and $\mathcal{F}_\Theta^- f(m)$ on $\mathfrak{a}_\mathbb{C}^*$ respectively defined for $\lambda \in \mathfrak{a}_\mathbb{C}^*$ by

$$\mathcal{F}_\Theta f(m;\lambda) := \frac{1}{|W_\Theta|}\int_{A_\Theta} f(a)\,\varphi_\Theta(m;\lambda,a)\mu(m;a)\,da = \int_{A^+} f(a)\varphi_\Theta(m;\lambda,a)\delta(m;a)\,da,$$
$$\mathcal{F}_\Theta^- f(m;\lambda) := \frac{1}{|W_\Theta|}\int_{A_\Theta} f(a)\,\varphi_\Theta^-(m;\lambda,a)\mu(m;a)\,da = \int_{A^+} f(a)\varphi_\Theta^-(m;\lambda,a)\delta(m;a)\,da,$$

where

$$\varphi_\Theta^-(m;\lambda,a) := \frac{\varphi_\Theta(m;\lambda,a)}{n_\Theta^-(m;\lambda)}.$$

We denote by $C_c(\overline{A_\Theta})^{W_\Theta}$ and $C_c^\infty(\overline{A_\Theta})^{W_\Theta}$ respectively the spaces of continuous and C^∞ W_Θ-invariant compactly supported functions on the closure $\overline{A_\Theta}$ of A_Θ.

Theorem 2.3.14 *Let $m \in \mathcal{M}^+$ and $\Theta \subset \Pi$.*

(a) *The Θ-spherical Fourier transform $\mathcal{F}_\Theta f(m;\lambda)$ of $f \in C_c(\overline{A_\Theta})^{W_\Theta}$ is a W_Θ-invariant meromorphic function of $\lambda \in \mathfrak{a}_\mathbb{C}^*$. Its singularities are located (counting multiplicities) in the polar set of the numerator function $n_\Theta^-(m;\lambda)$. When Σ is reduced, then all singularities are simple poles.*

(b) *The normalized Θ-spherical Fourier transform $\mathcal{F}_\Theta^- f(m;\lambda)$ of $f \in C_c(\overline{A_\Theta})^{W_\Theta}$ is a W_Θ-invariant entire function of $\lambda \in \mathfrak{a}_\mathbb{C}^*$.*

(c) *Suppose Σ is reduced and m is an even multiplicity function. Let $e_\Theta^-(m;\lambda)$ be the polynomial function of $\lambda \in \mathfrak{a}_\mathbb{C}^*$ defined in (1.57). Then $e_\Theta^-(m;\lambda)\mathcal{F}_\Theta f(m;\lambda)$ is a W_Θ-invariant entire function of $\lambda \in \mathfrak{a}_\mathbb{C}^*$ for all $f \in C_c(\overline{A_\Theta})^{W_\Theta}$.*

Proof. Part (a) is a consequence of (b), which we now prove. Let $R > 0$ be arbitrarily fixed. Recall from (2.46) that

$$\frac{1}{n_\Theta^-(m;\lambda)} = n_{\Theta,R}^-(m;\lambda)N_{\Theta,R}^-(m;\lambda),$$

where

$$N_{\Theta,R}^-(m;\lambda) := \left[\prod_{\alpha\in\Sigma_{\mathrm{i}}^+\setminus\langle\Theta\rangle_{\mathrm{i}}^+} \Gamma\left(-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}+\frac{1}{2}+R_{\alpha,1}\right)\Gamma\left(-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}-\frac{m_{2\alpha}}{2}+1+R_{\alpha,2}\right)\right]^{-1}$$

is W_Θ-invariant and holomorphic in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Observe that for the constant c occurring in Corollary 2.3.12 we have

$$\mu(m;a) = \mu(m;a)^{(c+1)/2}\mu(m;a)^{(1-c)/2}$$

with $(1-c)/2 \in (0,1/2]$.

Let $f \in C_c(\overline{A_\Theta})^{W_\Theta}$ and let $S > 0$ be chosen so that $\operatorname{supp} f \subset \{a \in \overline{A_\Theta} : |\log a| \leq S\} := B_S$. Applying the estimates (2.53), we obtain for $a \in A_\Theta$ and $\lambda \in \mathfrak{a}_{\mathbb{C},\Theta}^*(R)$

$$\begin{aligned}
&\left|f(a)\varphi_\Theta^-(m;\lambda,a)\right|\mu(m;a) = \\
&\quad = |f(a)|\left|n_{\Theta,R}^-(m;\lambda)\varphi_\Theta(m;\lambda,a)\right|\left|N_{\Theta,R}^-(m;\lambda)\right|\mu(m;a)^{(c+1)/2}\mu(m;a)^{(1-c)/2} \\
&\quad \leq C_{\Theta,R,S,c,m}\,\chi_S(a)\mu(m;a)^{(1-c)/2}\,\left|N_{\Theta,R}^-(m;\lambda)\right|(1+|\lambda|)^{d'(\Theta,R,m)}e^{S|\Re\lambda|},
\end{aligned}$$

where χ_S denotes the characteristic function of the set B_S. This estimate ensures that for fixed $\lambda \in \mathfrak{a}_{\mathbb{C},\Theta}^*(R)$, the integral defining $\mathcal{F}_\Theta^- f(m;\lambda)$ converges. Moreover, considered on a small neighborhood of each $\lambda \in \mathfrak{a}_{\mathbb{C},\Theta}^*(R)$, the inequality proves by the theorems of Fubini and Morera that $\mathcal{F}_\Theta^- f(m;\lambda)$ is holomorphic near λ. Thus $\mathcal{F}_\Theta^- f(m;\lambda)$ is holomorphic in λ on $\mathfrak{a}_{\mathbb{C},\Theta}^*(R)$. The claim of the Theorem then follows because the $\mathfrak{a}_{\mathbb{C},\Theta}^*(R)$ with $R > 0$ cover the entire $\mathfrak{a}_{\mathbb{C}}^*$.

Part (c) follows from part (b) together with the removability of singularities guaranteed by Theorem 1.6.6. □

The proof of Theorem 2.3.14 shows that in fact we can admit some singularity along the walls of A_Θ. We state this property only for the normalized Θ-spherical Fourier transform.

Corollary 2.3.15 *The normalized Θ-spherical Fourier transform $\mathcal{F}_\Theta^- f(m;\lambda)$ is a well-defined W_Θ-invariant entire function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ for all functions f satisfying $\mu(m;a)^{(1-c)/2} f \in C_c(\overline{A_\Theta})^{W_\Theta}$.* □

Remark 2.3.16 The estimates of Theorem 2.3.11 show that $\delta(m;a)^{(c+1)/2}\varphi_\Theta(m;\lambda,a)$ in nonsingular along the walls of A^+. Moreover, since for $A^+ \ni a \to +\infty$

$$\delta(m;a)^{(c+1)/2} \sim e^{(c+1)\rho(m)(\log a)},$$

they imply that, for a certain constant $C_{R,\lambda,m} > 0$,

$$\varphi_\Theta(m;\lambda,a) \lesssim C_{R,\lambda,m}e^{(\rho(m)+W_\Theta\lambda)(\log a)}$$

whenever $\lambda \in \mathfrak{a}_{\mathbb{C},\Theta}^*(R)$ and $A^+ \ni a \to +\infty$.

This suggests the possibility for the Θ-spherical Fourier transform of a Schwartz space theory in which, as in the Riemannian case, the correct definition of rapid decrease of a function f requires that for every $M \in \mathbb{N}$

$$f(a) = O\left(e^{-\rho(m)(\log a)}(1+|\log a|)^M\right)$$

for $a \in A^+$ (see e.g. [GV88], §6.1).

2.4 Inversion of the Θ-spherical Fourier transform

In this section we establish the inversion formula for the Θ-spherical Fourier transform on the space $C_c^\infty(A_\Theta)^{W_\Theta}$ of compactly supported W_Θ-invariant C^∞ functions on A_Θ. The inversion of the Θ-spherical Fourier transform can be obtained by generalizing the procedure followed by Ólafsson in the case of the spherical transform on noncompactly causal (NCC) symmetric spaces. The method employed in the NCC case is a reduction to the inversion of the Harish-Chandra's spherical transform on the dual Riemannian symmetric space. The method can be extended to the non-necessarily geometric case of triples $(\mathfrak{a}, \Sigma, m)$. Let Θ be a subset of the fundamental system Π of simple roots associated with the fixed positive system Σ^+. Then the inversion of the Θ-spherical Fourier transform will be deduced from the inversion of the Π-spherical Fourier transform. The inversion of the Π-spherical transform is due to Opdam, since this transform coincides (up to a constant multiple) with one of the transforms he studied in [Opd95].

The link between the Θ-spherical and Π-spherical Fourier transforms is given by Corollary 2.1.7, which is a generalization of the relation found by Ólafsson between the spherical transforms on a NCC space and its Riemannian dual.

As in the previous sections, we shall always assume for a multiplicity function m that $m \in \mathcal{M}^+$ and that $m_{2\alpha} = 0$ is equivalent to $2\alpha \notin \Sigma$.

Let $\Theta \subset \Pi$ be fixed. The following lemma rephrases in our notation some very well-known properties of the Harish-Chandra's c-function for imaginary values of the spectral parameter λ.

Lemma 2.4.1 *The functions $c_\Theta^\pm$ from Definition 1.4.1 satisfy the following properties:*

(a) $c_\Theta^\pm(m; -i\lambda) = \overline{c_\Theta^\pm(m; i\lambda)}$ *for all $\lambda \in \mathfrak{a}^*$, where $^-$ denotes complex conjugation;*

(b) $c_\Theta^\pm(m; \lambda)c_\Theta^\pm(m; -\lambda)$ *is W_Θ-invariant;*

(c) $|c_\Theta^\pm(m; i\lambda)|$ *is W_Θ-invariant for all $\lambda \in \mathfrak{a}^*$;*

(d) $1/c_\Theta^+(m; \lambda)$ *is a holomorphic function of λ in the open neighborhood*

$$B_{\Theta,m} := \left\{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \Re\lambda_\alpha > \max\{-m_\alpha/2 - 1, -m_\alpha/2 - m_{2\alpha}\} \quad \text{for all } \alpha \in \langle\Theta\rangle_{\mathrm{i}}^+\right\}$$

of $i\mathfrak{a}^*$ in $\mathfrak{a}_{\mathbb{C}}^*$.

(e) *There are constants $C_m > 0$ and $r(m) \geq 0$ such that*

$$\left|c_\Theta^+(m; -i\lambda)\right|^{-1} \leq C_m(1 + |\lambda|)^{r(m)}$$

for all $\lambda \in \mathfrak{a}^*$.

Proof. Part (a) is an immediate consequence of the definition of $c_\Theta^\pm$ and the property $\overline{\Gamma(z)} = \Gamma(\overline{z})$ of the gamma function.

To prove (b), it is enough to verify that $c_\Theta^+(m;\lambda)c_\Theta^+(m;-\lambda)$ is invariant under every reflection r_j associated with elements $\alpha_j \in \Theta$. Notice that (in the notation of Definition 1.4.1) we have $c_\alpha^+(m; r_j\lambda) = c_{r_j\alpha}^+(m;\lambda)$ and $c_{-\alpha}^+(m;\lambda) = c_\alpha^+(m;-\lambda)$. The claim follows then immediately from the remark that the reflection r_j permutes $\langle\Theta\rangle^+ \setminus \{\alpha_j, 2\alpha_j\}$ and maps α_j and $2\alpha_j$ into their opposites $-\alpha_j$ and $-2\alpha_j$, respectively.

We have for all $w \in W_\Theta$ and $\lambda \in \mathfrak{a}^*$

$$\begin{aligned}\left|c_\Theta^+(m; wi\lambda)\right|^2 &= c_\Theta^+(m; iw\lambda)\overline{c_\Theta^+(m; iw\lambda)} \\ &= c_\Theta^+(m; iw\lambda)c_\Theta^+(m; -iw\lambda) \qquad \text{(by (a))} \\ &= c_\Theta^+(m; i\lambda)c_\Theta^+(m; -i\lambda) \qquad \text{(by (b))} \\ &= \left|c_\Theta^+(m; i\lambda)\right|^2,\end{aligned}$$

which proves (c). Part (d) follows immediately from formula (1.37). Finally, part (e) is a consequence of the asymptotics (2.43) for ratios of gamma functions. □

Recall from (2.51) the notation a_Θ for the element of the W_Θ-orbit of $a \in A_\Theta$ lying in $\overline{A^+}$.

Lemma 2.4.2 *Let $m \in \mathcal{M}^+$, and let $c \in [0,1]$ be the constant occurring in the estimates of Theorem 2.2.18. Let B_Θ and $B_{\Theta,m}$ be the open neighborhoods of $i\mathfrak{a}^*$ in $\mathfrak{a}_\mathbb{C}^*$ introduced in Corollary 1.5.13 and in part (d) of Lemma 2.4.1. Define*

$$c_\Theta(m;\lambda) := c_\Theta^+(m;\lambda)c_\Theta^-(m;\lambda). \tag{2.58}$$

Then for every $a \in A_\Theta$ the function

$$\frac{\varphi_\Theta(m;\lambda,a)}{c_\Theta(m;\lambda)}$$

is holomorphic in $\lambda \in B_\Theta \cap B_{\Theta,m}$. Moreover, there is a constant $C_{m,c} > 0$ such that

$$\mu(m;a)\left|\frac{\varphi_\Theta(m;i\lambda,a)}{c_\Theta(m;i\lambda)}\right| \le C_{m,c}\,|W_\Theta|\,\mu(m;a)^{(1-c)/2}e^{c\rho(m)(\log a_\Theta)}$$

for all $a \in A_\Theta$ and $\lambda \in \mathfrak{a}^*$.

Proof. The statement on the holomorphy of $\varphi_\Theta(m;\lambda,a)/c_\Theta(m;\lambda)$ follows immediately from Corollary 1.5.13 and Lemma 2.4.1 (d).

According to (1.5.1), we have for all $a \in A^+$ and generic $\lambda \in \mathfrak{a}^*$

$$\frac{\varphi_\Theta(m;i\lambda,a)}{c_\Theta(m;i\lambda)} = \sum_{w\in W_\Theta}\frac{c_\Theta^+(m;iw\lambda)}{c_\Theta^+(m;i\lambda)}\Phi(m;iw\lambda,a).$$

Notice that $i\mathfrak{a}^* \subset \mathfrak{a}_{\mathbb{C}}^*(R)$ for all $R > 0$. We can therefore fix $R > 0$ small enough in Theorem 2.2.18 so that the polynomial p_R is identically equal to 1. Observing moreover that

$$\left| \frac{c_\Theta^+(m; iw\lambda)}{c_\Theta^+(m; i\lambda)} \right| = 1$$

by part (c) of Lemma 2.4.1, we obtain

$$\begin{aligned} \mu(m;a) \left| \frac{\varphi_\Theta(m; i\lambda, a)}{c_\Theta(m; i\lambda)} \right| &\leq \sum_{w \in W_\Theta} \mu(m;a) \, |\Phi(m; iw\lambda, a)| \\ &\leq C_{m,c} \sum_{w \in W_\Theta} \mu(m;a)^{(1-c)/2} e^{(c\rho(m) + \Re iw\lambda)(\log a)} \\ &= C_{m,c} \, |W_\Theta| \, \mu(m;a)^{(1-c)/2} e^{c\rho(m)(\log a_\Theta)} \end{aligned}$$

The inequality extends then by W_Θ-invariance to A_Θ and by continuity to all $\lambda \in \mathfrak{a}^*$. □

Corollary 2.4.3 *For every $f \in C_c^\infty(A_\Theta)^{W_\Theta}$, the function*

$$\frac{\mathcal{F}_\Theta f(m; i\lambda)}{c_\Theta(m; i\lambda)}$$

is real analytic in $\lambda \in \mathfrak{a}^*$. Moreover, for every $N \in \mathbb{N}$ there is a constant $C_{\Theta,N,m,c,f} > 0$ such that

$$\left| \frac{\mathcal{F}_\Theta f(m; i\lambda)}{c_\Theta(m; i\lambda)} \right| \leq C_{\Theta,N,m,c,f} (1 + |\lambda|)^{-N}$$

for all $\lambda \in \mathfrak{a}^*$.

Proof. For every $a \in A_\Theta$, the function $\varphi_\Theta(m; \lambda, a)/c_\Theta(m; \lambda)$ is holomorphic in $B_\Theta \cap B_{\Theta,m}$ and f is continuous and compactly supported in A_Θ. The usual application of the theorems of Fubini and Morera yields that $\mathcal{F}_\Theta f(m; \lambda)/c_\Theta(m; \lambda)$ is holomorphic in $\lambda \in B_\Theta \cap B_{\Theta,m}$. In particular, its restriction to $i\mathfrak{a}^*$ is real analytic.

As a consequence of Theorem 1.2.15, the operator $L(m)$ is self-adjoint with respect to the measure $\mu(m;a)\, da$ on A. Moreover for all $\lambda \in \mathfrak{a}^*$

$$\begin{aligned} L(m)\varphi_\Theta(m; i\lambda) &= -(\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle)\varphi_\Theta(m; i\lambda) \\ &= -\big(|\lambda|^2 + |\rho|^2 \big)\varphi_\Theta(m; i\lambda), \end{aligned}$$

which implies

$$L(m)^N \varphi_\Theta(m; i\lambda) = (-1)^N \big(|\lambda|^2 + |\rho|^2 \big)^N \varphi_\Theta(m; i\lambda)$$

for all $N \in \mathbb{N}$. The function f is C^∞ and compactly supported in A_Θ, so no boundary values occur in the following computation:

$$\begin{aligned} \big(|\lambda|^2 + |\rho|^2 \big)^N \frac{\mathcal{F}_\Theta f(m; i\lambda)}{c_\Theta(m; i\lambda)} &= \int_{A_\Theta} f(a) \big(|\lambda|^2 + |\rho|^2 \big)^N \frac{\varphi_\Theta(m; i\lambda, a)}{c_\Theta(m; i\lambda)} \, \mu(m;a) \, da \\ &= (-1)^N \int_{A_\Theta} f(a) \, \frac{L(m)^N \varphi_\Theta(m; i\lambda, a)}{c_\Theta(m; i\lambda)} \, \mu(m;a) \, da \\ &= (-1)^N \int_{A_\Theta} \big[L(m)^N f(a) \big] \, \frac{\varphi_\Theta(m; i\lambda, a)}{c_\Theta(m; i\lambda)} \, \mu(m;a) \, da \end{aligned}$$

Lemma 2.4.2 gives for some constant $C'_{\Theta,N,m,c,f} > 0$

$$\left(|\lambda|^2+|\rho|^2\right)^N \left|\frac{\mathcal{F}_\Theta f(m;i\lambda)}{c_\Theta(m;i\lambda)}\right| \le \int_{A_\Theta} \left|L(m)^N f(a)\right| \left|\frac{\varphi_\Theta(m;i\lambda,a)}{c_\Theta(m;i\lambda)}\right| \mu(m;a)\,da \le C'_{\Theta,N,m,c,f},$$

from which the claim follows easily. □

For every $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we define the $E_\Theta(m;\lambda)$ on A_Θ by requiring that the equality

$$E_\Theta(m;\lambda,a) = \frac{c_\Theta^-(m;\lambda)}{c_\Theta^{+,c}(m;\lambda)}\,\varphi_{\text{II}}(m;\lambda,a) \tag{2.59}$$

holds for all $a \in A^+$. Then $E_\Theta(m;\lambda,a)$ is a W_Θ-invariant real analytic function of $a \in A_\Theta$ and a W_Θ-invariant meromorphic function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

Remark 2.4.4 Notice that $E_{\text{II}}(m;\lambda,a) = \varphi_{\text{II}}(m;\lambda,a)$. Furthermore, when m is an even multiplicity function, we obtain from Lemma 1.4.5

$$\frac{c_\Theta^-(m;\lambda)}{c_\Theta^{+,c}(m;\lambda)} = \prod_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \frac{c_\alpha^-(m;\lambda)}{c_\alpha^+(m;\lambda)} = \prod_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \frac{\sin(\pi\lambda_\alpha)}{\sin\left[\pi(\lambda_\alpha+m_\alpha/2)\right]} = (-1)^{\frac{1}{2}\sum_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} m_\alpha},$$

that is $E_\Theta(m;\lambda,a) = \pm\varphi_{\text{II}}(m;\lambda,a)$ for all Θ.

Theorem 2.4.5 *There is a constant $k > 0$ (depending only on the normalization of the measures) so that for all $f \in C_c^\infty(A_\Theta)^{W_\Theta}$ the following inversion formula holds: For all $a \in A_\Theta$*

$$f(a) = k\frac{|W|}{|W_\Theta|}\int_{i\mathfrak{a}^*} \mathcal{F}_\Theta f(m;\lambda)\,E_\Theta(m;-\lambda,a)\,\frac{d\lambda}{\left|c_\Theta(m;\lambda)\right|^2}, \tag{2.60}$$

where $d\lambda$ is a suitable normalization of the Lebesgue measure on $i\mathfrak{a}^*$.

Proof. Let $f \in C_c^\infty(A_\Theta)^{W_\Theta}$ and extend it by W-invariance to a function in $C_c^\infty(A)$, which we still denote by the symbol f. Opdam's inversion formula for $\mathcal{F}_{\text{II}}(m)$ gives for all $a \in A$ and for a constant $k > 0$ depending only on the normalization of the measures

$$f(a) = k\int_{i\mathfrak{a}^*} \mathcal{F}_{\text{II}} f(m;\lambda)\,\varphi_{\text{II}}(m;-\lambda,a)\,\frac{d\lambda}{\left|c_{\text{II}}^+(m;\lambda)\right|^2}$$

(see Theorem 9.13(3) in [Opd95] or formula (8.7) in [Opd00]). The functional equation (2.6), the W-invariance of $\varphi_{\text{II}}(m;\lambda,a)$ and $c_{\text{II}}^+(m;\lambda)$, and the properties of the functions $c_\Theta^\pm$

from Lemma 2.4.1 imply therefore the following formal computation for $a \in A^+$:

$$\begin{aligned}
f(a) &= k \int_{i\mathfrak{a}^*} \mathcal{F}_\Pi f(m;\lambda)\, \varphi_\Pi(m;-\lambda,a)\, \frac{d\lambda}{\left|c_\Pi^+(m;\lambda)\right|^2} \\
&= k \sum_{w\in W_\Theta\backslash W} \int_{i\mathfrak{a}^*} \mathcal{F}_\Theta f(m;w\lambda)\, \varphi_\Pi(m;-\lambda,a)\, \frac{c_\Theta^{+,c}(m;w\lambda)}{c_\Theta^-(m;w\lambda)}\, \frac{d\lambda}{c_\Pi^+(m;\lambda)c_\Pi^+(m;-\lambda)} \\
&= k \sum_{w\in W_\Theta\backslash W} \int_{i\mathfrak{a}^*} \mathcal{F}_\Theta f(m;w\lambda) \frac{\varphi_\Pi(m;-w\lambda,a)}{c_\Theta^-(m;w\lambda)c_\Theta^{+,c}(m;-w\lambda)}\, \frac{d\lambda}{c_\Theta^+(m;w\lambda)c_\Theta^+(m;-w\lambda)} \\
&= k \sum_{w\in W_\Theta\backslash W} \int_{i\mathfrak{a}^*} \mathcal{F}_\Theta f(m;\lambda) \frac{\varphi_\Pi(m;-\lambda,a)}{c_\Theta^-(m;\lambda)c_\Theta^{+,c}(m;-\lambda)}\, \frac{d\lambda}{c_\Theta^+(m;\lambda)c_\Theta^+(m;-\lambda)} \\
&= k\, \frac{|W|}{|W_\Theta|} \int_{i\mathfrak{a}^*} \mathcal{F}_\Theta f(m;\lambda)\, E_\Theta(m;-\lambda,a)\, \frac{d\lambda}{|c_\Theta(m;\lambda)|^2}.
\end{aligned}$$

The computation is justified provided at the first step we can interchange the integral sign with the summation. This is equivalent to verifying the convergence of the integral

$$\int_{i\mathfrak{a}^*} \mathcal{F}_\Theta f(m;\lambda)\, E_\Theta(m;-\lambda,a)\, \frac{d\lambda}{|c_\Theta(m;\lambda)|^2}.$$

The absolute value of the integrand is

$$\left|\frac{\mathcal{F}_\Theta f(m;\lambda)}{c_\Theta(m;\lambda)}\right| \left|\frac{c_\Theta^-(m;-\lambda)\varphi_\Pi(m;-\lambda,a)}{c_\Theta^{+,c}(m;-\lambda)c_\Theta(m;-\lambda)}\right| = \left|\frac{\mathcal{F}_\Theta f(m;\lambda)}{c_\Theta(m;\lambda)}\right| \left|\frac{\varphi_\Pi(m;-\lambda,a)}{c_\Pi^+(m;-\lambda)}\right|.$$

Because of part (e) in Lemma 2.4.1, there are constants $C > 0$ and $r \geq 0$ (depending on m) such that

$$\left|c_\Pi^+(m;-\lambda)\right|^{-1} \leq C(1+|\lambda|)^r$$

for all $\lambda \in i\mathfrak{a}^*$. Opdam's estimates (2.55) prove the existence of a constant C' (depending on m) such that

$$|\varphi_\Pi(m;-\lambda,a)| \leq C'$$

for all $\lambda \in i\mathfrak{a}^*$ and $a \in A_\Theta$ Finally, Corollary 2.4.3 yields for all $\lambda \in i\mathfrak{a}^*$ and all $n \in \mathbb{N}$

$$\left|\frac{\mathcal{F}_\Theta f(m;\lambda)}{c_\Theta(m;\lambda)}\right| \leq C_{\Theta,N,m,c,f}(1+|\lambda|)^{-N}.$$

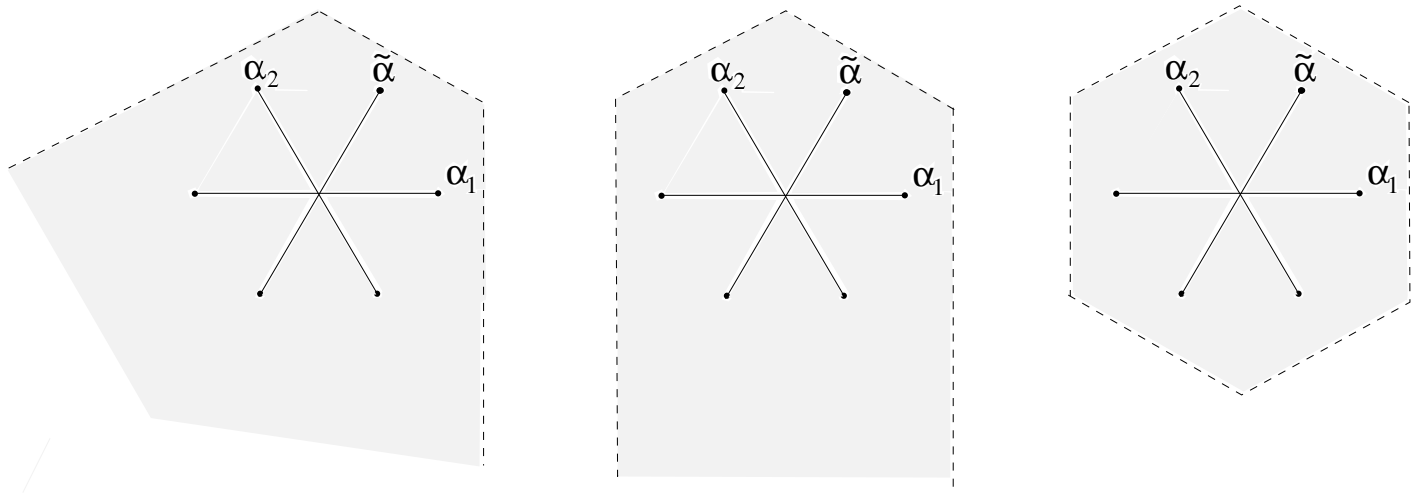
By selecting $N \in \mathbb{N}$ large enough, the required convergence is achieved. □

Remark 2.4.6

(a) Observe that in the proof of Theorem 2.4.5 we had to employ Opdam's estimate (2.55) for the function φ_Π. Because of the superfluous term $\mu(m;a)^{(1-c)/2}$, our estimates from Theorem 2.3.11 would have guaranteed the convergence of the integral on A^+ (and hence on $W_\Theta(A^+)$ by W_Θ-invariance), but not on the entire A_Θ.

(b) Since $\varphi_\Pi(m;\lambda,a) = E_\Pi(m;\lambda,a)$, the version of the inversion formula given by Theorem 2.4.5 is the exact analogue for the set $\Theta \subset \Pi$ of Opdam's inversion formula for the

transform associated with the hypergeometric functions associated with root systems. Up to a constant multiple, it agrees with Opdam's inversion formula for $\Theta = \Pi$, and hence with the inversion formula for the spherical transform on Riemannian symmetric spaces in the geometric case (see e.g. [GV88], Theorem 6.4.1). In the geometric case of noncompactly causal symmetric spaces, the inversion formula reduces to Ólafsson's formula (see Theorem 6.13 in [Ól97] or Theorem 8.5.3 in [HÓ96]).



(a) $\Theta = \emptyset$ (b) $\Theta = \{\alpha_1\}$ (c) $\Theta = \Pi$

Figure 2.5: The real part of the domain $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ for Σ of type A_2, with $\Sigma^+ = \{\alpha_1, \alpha_2, \widetilde{\alpha} := \alpha_1 + \alpha_2\}$, $\Pi = \{\alpha_1, \alpha_2\}$

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