ASYMPTOTIC ANALYSIS OF Θ-HYPERGEOMETRIC FUNCTIONS

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Abstract. We define the Θ-hypergeometric functions as a generalization of the hypergeometric functions associated with root systems of Heckman and Opdam. In the geometric setting, the Θ-hypergeometric functions can be specialized to Harish-Chandra's spherical functions on Riemannian symmetric spaces of noncompact type, and also to the spherical functions on noncompactly causal symmetric spaces. After describing their regularity properties, we prove estimates for the Θ-hypergeometric functions which are uniform in the space parameter and locally uniform in the spectral parameter. Particular cases are sharp uniform estimates for the Harish-Chandra series up to the walls of the positive Weyl chamber. New estimates for the spherical functions on noncompactly causal symmetric spaces are deduced.

Introduction

The Θ-hypergeometric functions originate from the harmonic analysis on symmetric spaces, more precisely, from the theory of multivariable hypergeometric functions of Heckman and Opdam and from the analysis of spherical distributions. In the late 1980s Heckman and Opdam defined the hypergeometric functions associated with root systems as multivariable analogue of the Jacobi functions of first kind (see [17], [15], [31] and [32]). The foundation of their construction was the interrelation between the theory of special functions in one variable and the harmonic analysis on rank-one Riemannian symmetric spaces G/K of noncompact type. Indeed, for special values of their parameters, the Jacobi functions of first kind agree with the restriction to a maximal flat subspace of the spherical functions on G/K. Proceeding in the opposite direction, Heckman and Opdam defined Jacobi functions of first kind in more variables by a suitable analytic continuation of the spherical functions on higher rank Riemannian symmetric spaces of noncompact type. From the point of view of harmonic analysis, this is not a mere generalization of Harish-Chandra's theory: by constructing the most appropriate context in which the spherical functions on Riemannian symmetric spaces can be studied by spectral-theoretic methods, Heckman and Opdam could develop very powerful techniques (e.g. monodromy techniques and the methods of shift and of analytic continuation in the spectral parameters) which allow to understand Harish-Chandra's spherical functions without using integral formulas. It is therefore quite natural to try to generalize the theory of Heckman and Opdam for the harmonic analysis of H-spherical distributions on a pseudo-Riemannian symmetric spaces G/H, where integral formulas are generally not available (see e.g. [25] and references therein). The theory of Θ-hypergeometric functions is such a generalization: it allows a successful analysis of certain spherical distributions on the so-called noncompactly causal symmetric spaces and provides a potential framework for studying spherical distributions on the larger class of K_ϵ-spaces of Oshima and Sekiguchi [37].

The definition of the Θ-hypergeometric functions is suggested by the theory of spherical functions on noncompactly causal (NCC) symmetric spaces, in particular by Ólafsson's expansion formula [26]. The NCC symmetric spaces build a specific class of pseudo-Riemannian symmetric spaces whose study started in connection with the global properties of the space-time (see e.g. [13]). The fine structure of a NCC symmetric space G/H (characterized by the existence of a certain

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Ad(H)-invariant convex cone in the tangent space at the base point $\{H\}$) allowed Faraut, Hilgert and Ólafsson to extend in 1994 the theory of spherical functions of Harish-Chandra to this non-Riemannian context (see [7], [20] and [26]). The spherical functions on G/H are the restrictions of certain H-spherical distributions associated with the principal series representations of G to the interior of a specific submanifold S/H described by the causal structure. Similarly to the Riemannian case, the spherical functions on G/H are H-invariant joint eigenfunctions of the commutative algebra of G-invariant differential operators on G/H. In [26], Ólafsson could relate them to the eigenfunctions of the algebra of G-invariant differential operators on the dual Riemannian symmetric space to G/H, which is isomorphic to G/K. This allowed him to prove his expansion formula, which expresses the spherical functions on G/H as a linear combination of the Harish-Chandra series of G/K. Since the starting point of the theory of Heckman and Opdam is the expansion of spherical functions on Riemannian symmetric spaces as linear combination of Harish-Chandra series, Ólafsson's formula indicates precisely how to generalize the hypergeometric functions of Heckman and Opdam to include, together with the spherical functions on Riemannian symmetric spaces, also the spherical functions on NCC symmetric spaces as geometric special instances.

Despite the analogies with the Riemannian case, the non-compactness of H causes much less regularity for the spherical functions on a NCC symmetric space G/H: they are only defined on the interior of the submanifold S/H and generally singular on its boundary $\partial(S/H)$; they are meromorphic (not entire) as functions of the spectral parameter; they can be described by integral formulas only for special values of the spectral parameter (hence these integral formulas cannot be used for global estimates); the associated spherical Fourier transform, which is defined by integration on S/H, can a priori only be applied to functions with compact support in S/H to compensate the singularities of the spherical functions on $\partial(S/H)$. The theory developed in this paper provides an efficient (and today the only) way to handle or solve all these problems. In particular, it determines the exact nature and location of the singularities in the spectral parameter and gives the first global estimates for these spherical functions. See Theorem 3.5 and Example 3.7, Corollary 5.6 and Proposition 5.9 below; see also Theorem 8.7 in [28].

The general context of special functions associated with root systems allows us to employ and adapt to the Θ-hypergeometric functions the powerful methods of Heckman and Opdam. The results obtained in this general setting can then be applied to the various special instances, e.g. to the spherical functions on NCC symmetric spaces. On the other hand, the spherical functions on NCC symmetric spaces justify the introduction of the Θ-hypergeometric functions as a natural class of special functions in more variables. In fact, for special values of their parameters, the Jacobi functions of second type agree with the restriction to a Weyl chamber in a maximal flat subspace of the spherical functions on rank-one NCC symmetric spaces. Containing as special cases both the spherical functions on Riemannian symmetric spaces of noncompact type and the spherical functions on NCC symmetric spaces, the Θ-hypergeometric functions are not only useful extensions of the hypergeometric functions of Heckman and Opdam, but also geometrically-motivated multivariable analogues of the Jacobi functions (cf. [10] and [16]). The theory of Θ-hypergeometric functions will be therefore developed in this paper from the point of view of special functions associated with root systems.

The parameter Θ designates a subset of a fixed fundamental system Π of positive simple roots in a given root system Σ. The different choices for Θ lead to a lattice of special functions associated with the root system. At the top of the lattice are the hypergeometric functions of Heckman and Opdam, which correspond to $\Theta = \Pi$, and at the bottom (certain multiples of) the Harish-Chandra series, which correspond to $\Theta = \emptyset$. In the middle there appear many new special functions. When Σ is the restricted root system corresponding to a NCC symmetric space G/H and with Θ equal to the set of positive compact simple roots, we obtain special functions which include the spherical functions on G/H. In the rank-one case there are only two choices for Θ, and they respectively lead to the Jacobi functions of first and second kind.

Suppose Σ is a root system in the dual space $\mathfrak{a}^*$ of a Euclidean space $\mathfrak{a}$. Let W denote the Weyl group of Σ. A multiplicity function on Σ is a W-invariant complex-valued function m on Σ. For a fixed Θ, the Θ-hypergeometric functions $\varphi_\Theta(m;\lambda,a)$ depend on three parameters: a multiplicity function m; a spectral parameter λ in the complex dual $\mathfrak{a}_\mathbb{C}^*$ of $\mathfrak{a}$; a space parameter a in a certain open domain A_Θ in a split Cartan space A with Lie algebra $\mathfrak{a}$. One has $A_\Theta = \exp\mathfrak{a}_\Theta$, where $\mathfrak{a}_\Theta$ is the smallest open convex cone containing the positive Weyl chamber $\mathfrak{a}^+$ and invariant under the Weyl group W_Θ of the root system generated by Θ. We refer to Section 3 for the precise definitions.

For "generic" $\lambda\in\mathbb{C}$ the Jacobi functions of second kind with spectral parameter λ and $-\lambda$ form a basis for the solution space of the Jacobi differential equation on $(0,+\infty)$. The Jacobi functions of first kind with spectral parameter λ can be therefore written on $(0,+\infty)$ as their linear combination by means of the classical transition relations (see e.g. [6], 2.9(34)). The coefficients are functions of λ given by products of ratios of gamma functions. Similarly, for a fixed root system Σ, for a fixed multiplicity function m on Σ and for "generic" values of $\lambda\in\mathfrak{a}_\mathbb{C}^*$, each Θ-hypergeometric function of spectral parameter λ is obtained on the positive Weyl chamber $A^+ := \exp\mathfrak{a}^+$ as a linear combination of the Harish-Chandra series of spectral parameter $w\lambda$ for w in W_Θ:

$$\varphi_\Theta(m;\lambda,a)=\sum_{w\in W_\Theta} c_\Theta(m;w\lambda)\Phi(m;w\lambda,a),\qquad a\in A^+, \tag{1}$$

where $\Phi(m;\lambda,a)$ denotes the Harish-Chandra series of spectral parameter λ. The coefficients occurring in (1) are obtained from the product $c_\Theta(m;\lambda):=c_\Theta^+(m;\lambda)c_\Theta^-(m;\lambda)$ of a function c_Θ^+, which is modeled on the Harish-Chandra c-function, and of a function c_Θ^-, which is modeled on the function c_Ω of Krötz and Ólafsson [24]. The functions c_Θ are therefore products of ratios of gamma functions depending on m, λ and on the root structure. Motivating the given definition is the requirement for the Θ-hypergeometric functions to generalize the spherical functions on symmetric spaces. Indeed, suppose m comes from the root multiplicities of a Riemannian symmetric spaces of noncompact type (such m's will be called geometric; see Section 1). Then, for suitable choices of Θ, the right-hand side of (1) specializes to Harish-Chandra's expansion formula for the spherical functions on Riemannian symmetric spaces of noncompact type and to Ólafsson's expansion formula for the spherical functions on NCC symmetric spaces. To underline their geometric origin, we denote the Θ-hypergeometric functions corresponding to geometric multiplicities as Θ-spherical functions.[1] It follows also from their definition that all Θ-hypergeometric functions of spectral parameter λ are joint eigenfunctions of the hypergeometric system of differential equations of spectral parameter λ constructed by Heckman-Opdam-Cherednik (cf. Section 1.1).

The regularity properties of the Θ-hypergeometric functions increase with Θ. The hypergeometric functions of Heckman and Opdam are holomorphic in a suitable tubular neighborhood of A in its complexification $A_\mathbb{C}$ and entire in the spectral parameter. At the other extreme of the Θ-lattice we have the Harish-Chandra series. They are generally only defined and holomorphic in a tubular neighborhood of A^+ in $A_\mathbb{C}$ and they are meromorphic in the spectral parameter, with singularities along complex hyperplanes associated with all roots. The Harish-Chandra series can be directly investigated by means of their defining recursion relations. Information on the Θ-hypergeometric functions for arbitrary Θ (and in particular for the spherical functions on NCC symmetric spaces) can be obtained moving "upwards" from the Harish-Chandra series by means of (1), or moving "downwards" from the hypergeometric functions of Heckman and Opdam (that is, "averaging" the Θ-hypergeometric functions to get the hypergeometric functions of Heckman and Opdam). The Θ-hypergeometric functions turn out to be holomorphic in a tubular neighborhood of the cone A_Θ in $A_\mathbb{C}$. As functions of the spectral parameter, they are meromorphic, with singularities located along complex hyperplanes associated with the roots in Σ which are not in the root subsystem generated by Θ. The locus of possible singularities and their order can be described by means of the "numerator" of the function c_Θ^-. In the multiplicity variable m there will be no singularities provided a suitable normalization of the Θ-hypergeometric functions is chosen. See Theorem 3.5.

The estimates of the Θ-hypergeometric functions form the core of this paper. A first group of estimates is treated in Section 4, where we estimate the Harish-Chandra series. The method

[1]The employ of the name Θ-spherical functions for the geometric case and Θ-hypergeometric functions for the general case was kindly suggested by the referee.

employed has its roots in the celebrated estimates of Gangolli [8]. But various modifications are introduced, and we also employ ideas of Wallach [42] and of Van den Ban and Schlichtkrull [3]. The resulting estimates, stated in Theorem 4.17, are quite sharp. They are uniform in $a \in A^+$ and locally uniform in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Moreover, they are in the context of special functions associated with root systems, i.e. they hold for every (not necessarily geometric) positive multiplicity function. There are two new important features of these estimates: first, they prove that for every positive multiplicity function m and for generic $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ the function $\Phi(m;\lambda,a)\delta(m;a)$ vanishes on the "walls" of A^+, where

$$\delta(m) := \prod_{\alpha\in\Sigma^+} \left|e^{\alpha} - e^{-\alpha}\right|^{m_\alpha}; \tag{2}$$

second, the domains in the λ-variables on which the estimates are uniform contain W-invariant domains which cover the entire $\mathfrak{a}_{\mathbb{C}}^*$. This is crucial to get estimates for $\varphi_\Theta(m;\lambda,a)$ for *all* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ by means of the estimates for the $\Phi(m;\lambda,a)$ and the defining formula (1) of $\varphi_\Theta(m;\lambda,a)$. The procedure is developed in Section 5. The resulting estimates for the Θ-hypergeometric functions are stated in Theorem 5.5. They imply that $\varphi_\Theta(m;\lambda,a)\delta(m;a)$ vanishes along the "walls" of A_Θ. From Theorem 5.5 we derive new estimates for the spherical functions on the NCC symmetric spaces.

A noteworthy application of our estimates concerns the spherical Fourier transform. The spherical transform on Riemannian and NCC symmetric spaces can be easily generalized to define a Θ-hypergeometric Fourier transform. For a positive multiplicity function m, the Θ-hypergeometric Fourier transform of a W_Θ-invariant function f on A_Θ is the W_Θ-invariant function $\mathcal{F}_\Theta f(m)$ on $\mathfrak{a}_{\mathbb{C}}^*$ defined for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ by

$$\mathcal{F}_\Theta f(m;\lambda) := \frac{1}{|W_\Theta|} \int_{A_\Theta} f(a)\, \varphi_\Theta(m;\lambda,a)\delta(m;a)\; da \tag{3}$$

provided the integral converges. Here da is a normalization of the Haar measure on A. When $\Theta = \Pi$, the transform has been studied by Opdam in [33]. In this case, as in the Riemannian case, the convergence of the integral is guaranteed by a sufficiently fast decay at infinity. When $\Theta \neq \Pi$ the situation is more complicated because one also has to take into account the behavior of the integrand along the walls of A_Θ, where the Θ-hypergeometric functions present singularities. The existence of these singularities is well-known in the case $\Theta = \emptyset$ (when the Θ-hypergeometric functions are essentially the Harish-Chandra series) and also in the geometric case of spherical functions on the NCC symmetric spaces. However, the nature and order of the singularities is generally not known. For this reason, all works on the harmonic analysis on NCC symmetric spaces have so far considered only the spherical transform of functions which are compactly supported in A_Θ, in particular, which stay far away from the walls of A_Θ. This is too restrictive. One would like to work on more structured spaces, such as L^p-spaces or Schwartz spaces. The problem is overcome once one knows that $\varphi_\Theta(m;\lambda,a)\delta(m;a)$ vanishes along the walls of A_Θ. Our estimates are therefore an important initial step in the harmonic analysis of Θ-hypergeometric functions, and in particular for the harmonic analysis on NCC symmetric spaces. Our estimates also allow us to obtain an inversion formula for the Θ-hypergeometric transform with Θ arbitrary as a consequence of the one proven by Opdam in the case $\Theta = \Pi$. For W_Θ-invariant compactly supported smooth functions, the inversion formula is given for $a \in A_\Theta$ by

$$f(a) = k\frac{|W|}{|W_\Theta|} \int_{i\mathfrak{a}^*} \mathcal{F}_\Theta f(m;\lambda)\, E_\Theta(m;-\lambda,a)\; \frac{d\lambda}{|c_\Theta(m;\lambda)|^2}\,.$$

Here $E_\Theta(m;\lambda)$ is the W_Θ-invariant real analytic function on A_Θ defined by requiring that the equality

$$E_\Theta(m;\lambda,a) = \frac{c_\Theta(m;\lambda)}{c_\Pi(m;\lambda)}\, \varphi_\Pi(m;\lambda,a)$$

holds for all $a \in A^+$. Moreover, $d\lambda$ is a suitable normalization of the Lebesgue measure on $i\mathfrak{a}^*$, and k is a positive constant depending only on the normalization of the measures. See [39].

Further applications of our estimates concern the case of even multiplicity functions, studied by Ólafsson and the author in [30]. In the context of Riemannian or NCC symmetric spaces G/H, even multiplicity functions correspond to the assumption that the Lie algebra of G possesses a unique conjugacy class of Cartan subalgebras. When the multiplicity function is even, each Harish-Chandra series $\Phi(m;\lambda,a)$ can be obtained on A^+ from the exponential function $e^{\lambda(\log a)}$ by means of a differential operator $D(m)$ with singular coefficients. $D(m)$ is one of Opdam's shift operators (see e.g. [14], Chapter 3). Let $\Delta(m) := \prod_{\alpha\in\Sigma^+}(e^\alpha - e^{-\alpha})^{m_\alpha}$. Our estimates for Φ are the essential tool in the proof that $\Delta(m)D(m)$ extends as a W-invariant differential operators with holomorphic coefficients on a tubular neighborhood of A in $A_{\mathbb{C}}$. This has several consequences: an elementary independent proof of Opdam's results in the case $\Theta = \Pi$; Paley-Wiener type theorems for the Θ-hypergeometric transform; explicit formulas for the Θ-hypergeometric functions. We point out that we obtain, as special instances, new explicit formulas for the spherical functions on Riemannian symmetric spaces of noncompact type with even multiplicities. We refer to the forthcoming publications [30] and [4] for further information.

The present work has to be considered as part of a larger project dealing with the harmonic analysis of the Θ-hypergeometric functions. There are several motivations for proceeding in this direction. As already mentioned, the Θ-hypergeometric functions represent a natural generalization of the hypergeometric functions of Heckman and Opdam and their primary application is the study of the spherical functions on NCC symmetric spaces. The regularity properties, on which this paper focuses, are only the first step for the harmonic analysis. Most of the analysis of spherical functions on NCC symmetric spaces is either in construction or yet undone: the L^2 and Schwartz space theory have never been studied, and Paley-Wiener theorems have been proven only in special cases (rank-one and even multiplicity cases, and some additional special series; see [1], [2], [30], and references therein). We believe that, as well as the theory of Heckman and Opdam did for the theory of spherical functions on Riemannian symmetric spaces, the approach by means of Θ-hypergeometric functions can clarify many of the parts of harmonic analysis on NCC symmetric spaces which are at this point still obscure. Another reason is related to the different choices for the set of simple roots Θ. Riemannian symmetric spaces of the noncompact type and NCC symmetric spaces are special instances of K_ϵ-symmetric spaces according to Oshima and Sekiguchi. The Θ-hypergeometric functions give a unified approach to the spherical functions and distributions on Riemannian and NCC symmetric spaces. It is an interesting and challenging problem to try to relate them also to spherical distributions on K_ϵ-spaces. The potential link is underlined by a natural bijection between the set of possible signatures ϵ for these symmetric spaces and the possible choices of Θ for the Θ-hypergeometric functions. It seems not accidental that the trivial signature (i.e. the signature of the Riemannian symmetric spaces) corresponds to $\Theta = \Pi$ and the signature for the NCC symmetric spaces corresponds to the set of positive compact simple roots. Further hints of a relation come from the study of the invariant differential operators in [37], Section 2.3. A final motivation comes from the study of distribution characters of discrete series representations. Despite the enormous advances in the harmonic analysis occurred in the last few years, explicit formulas for the distribution characters of discrete series representation is in most cases still missing. We refer the reader to Section 5 in [27] for a survey. In the group case this formula is due to Harish-Chandra [11]. For the holomorphic discrete series it was proven that the H-distribution characters can be realized as hyperfunctions using the spherical functions on NCC symmetric spaces G/H (see [23] and [27]). It is therefore an interesting problem if the Θ-hypergeometric functions can be related to the H-distribution characters of other discrete series representations. In conclusion, the investigation of the precise role played by the Θ-hypergeometric functions in the harmonic analysis on symmetric spaces is, in our opinion, an important topic and a ground for further studies in the direction initiated by the present paper.

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1. Notation and preliminaries

Let $\mathfrak{a}$ be an l-dimensional real Euclidean vector space with inner product $\langle \cdot, \cdot \rangle$. For every α in the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$, let $A_\alpha \in \mathfrak{a}$ be determined by the condition that $\alpha(H) = \langle H, A_\alpha \rangle$ for all $H \in \mathfrak{a}$. Hence $H_\alpha := 2A_\alpha/\langle A_\alpha, A_\alpha \rangle$ satisfies $\alpha(H_\alpha) = 2$. The assignment $\langle \alpha, \beta \rangle := \langle A_\alpha, A_\beta \rangle$ defines an inner product in $\mathfrak{a}^*$. Let Σ be a (possibly nonreduced) root system in the dual $\mathfrak{a}^*$ with associated Weyl group W. For every $\alpha \in \Sigma$, we denote by r_α the reflection in $\mathfrak{a}^*$ defined by $r_\alpha(\lambda) := \lambda - \lambda(H_\alpha)\alpha$ for all $\lambda \in \mathfrak{a}^*$.

Let Σ^+ be a choice of positive roots in Σ and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ the fundamental system of simple roots associated with Σ^+. The positive Weyl chamber $\mathfrak{a}^+$ consists of the elements $H \in \mathfrak{a}$ for which $\alpha(H) > 0$ for all $\alpha \in \Sigma^+$. We denote by Σ_{i} and Σ_{u} the indivisible and unmultipliable roots in Σ, respectively. Their set of positive elements are denoted by Σ_{i}^+ and Σ_{u}^+.

The complexification $\mathfrak{a}_\mathbb{C} := \mathfrak{a} \otimes_\mathbb{R} \mathbb{C}$ of $\mathfrak{a}$ can be viewed as the Lie algebra of the complex torus $A_\mathbb{C} := \mathfrak{a}_\mathbb{C}/\mathbb{Z}\{i\pi H_\alpha : \alpha \in \Sigma\}$. We write $\exp : \mathfrak{a}_\mathbb{C} \to A_\mathbb{C}$ for the exponential map, with multi-valued inverse log. The split real form $A := \exp \mathfrak{a}$ of $A_\mathbb{C}$ is an abelian subgroup of $A_\mathbb{C}$ with Lie algebra $\mathfrak{a}$ such that $\exp : \mathfrak{a} \to A$ is a diffeomorphism. We set $A^+ := \exp \mathfrak{a}^+$. The polar decomposition of $A_\mathbb{C}$ is $A_\mathbb{C} = AT$, where $T := \exp(i\mathfrak{a})$ is a compact torus with Lie algebra $i\mathfrak{a}$. Let $\mathfrak{a}_\mathbb{C}^*$ be the space of all $\mathbb{C}$-linear functionals on $\mathfrak{a}$. The action of W extends to $\mathfrak{a}$ by duality, to $\mathfrak{a}_\mathbb{C}^*$ and $\mathfrak{a}_\mathbb{C}$ by $\mathbb{C}$-linearity, and to $A_\mathbb{C}$ and A by the exponential map. Moreover, W acts on functions f on any of these spaces by $(wf)(x) := f(w^{-1}x)$, $w \in W$. The $\mathbb{C}$-bilinear extension to $\mathfrak{a}_\mathbb{C}^*$ and $\mathfrak{a}_\mathbb{C}$ of the inner products $\langle \cdot, \cdot \rangle$ on $\mathfrak{a}^*$ and $\mathfrak{a}$ will also be denoted by $\langle \cdot, \cdot \rangle$.

A multiplicity function on Σ is a W-invariant function $m : \Sigma \to \mathbb{C}$. Setting $m_\alpha := m(\alpha)$ for $\alpha \in \Sigma$, we therefore have $m_{w\alpha} = m_\alpha$ for all $w \in W$. We extend m to $\mathfrak{a}^*$ by putting $m_\alpha = 0$ for $\alpha \notin \Sigma$. We say that a multiplicity function m is *geometric* if there is a Riemannian symmetric space of noncompact type G/K with restricted root system Σ such that m_α is the multiplicity of the root α for all $\alpha \in \Sigma$. Otherwise, m is said to be *non-geometric.* [2] The set $\mathcal{M}$ of all multiplicity functions on Σ is a subspace of the finite-dimensional $\mathbb{C}$-vector space $\mathbb{C}^\Sigma$. The real subspace of $\mathcal{M}$ consisting of all multiplicity functions m with $m_\alpha \geq 0$ for all $\alpha \in \Sigma$ is denoted by $\mathcal{M}^+$. The complex dimension $d(\mathcal{M})$ of $\mathcal{M}$ equals the number of W-orbits in Σ. Thus $\mathcal{M}$ can be treated analytically as $\mathbb{C}^{d(\mathcal{M})}$.

The dimension l of $\mathfrak{a}$ will also be called the *(real) rank* of the triple $(\mathfrak{a}, \Sigma, m)$.

For $\alpha \in \Sigma$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$ we set

$$\lambda_\alpha := \frac{\lambda(H_\alpha)}{2} = \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}. \tag{4}$$

Define

$$P := \{\lambda \in \mathfrak{a}^* : \lambda_\alpha \in \mathbb{Z} \quad \text{for all } \alpha \in \Sigma\},$$
$$P^+ := \{\lambda \in \mathfrak{a}^* : \lambda_\alpha \in \mathbb{N}_0 \quad \text{for all } \alpha \in \Sigma^+\}.$$

Here and in the following we adopt the conventions $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

We call P the *restricted weight lattice* of Σ. The elements of P^+ are the *dominant restricted weights*. Observe that $\{2\alpha : \alpha \in \Sigma\} \subset P$. When m is geometric, then P^+ coincides with the set of restrictions to $\mathfrak{a}$ of the highest weights of the finite-dimensional irreducible K-spherical representations of G (see [19], Theorem 4.1, p. 535).

If $\lambda \in P$, then the exponential $e^\lambda : A_\mathbb{C} \to \mathbb{C}$ given by $e^\lambda(h) := e^{\lambda(\log h)}$ is single valued. The e^λ are the algebraic characters of $A_\mathbb{C}$. Their $\mathbb{C}$-linear span coincides with the ring of regular functions $\mathbb{C}[A_\mathbb{C}]$ on the affine algebraic variety $A_\mathbb{C}$. The lattice P is W-invariant, and the Weyl group acts on $\mathbb{C}[A_\mathbb{C}]$ according to $w(e^\lambda) := e^{w\lambda}$. The set $A_\mathbb{C}^{\mathrm{reg}} := \{h \in A_\mathbb{C} : e^{2\alpha(\log h)} \neq 1 \text{ for all } \alpha \in \Sigma\}$ consists of the regular points of $A_\mathbb{C}$ for the action of W. Notice that A^+ is a subset of $A_\mathbb{C}^{\mathrm{reg}}$. The algebra

[2] We adopt the multiplicity notation commonly used in the theory of symmetric spaces. It differs from the notation employed by Heckman and Opdam in the following ways. The root system R used by Heckman and Opdam is related to our root system Σ by the relation $R = \{2\alpha : \alpha \in \Sigma\}$; the multiplicity function k in Heckman-Opdam's work is related to our m by $k_{2\alpha} = m_\alpha/2$.

$\mathbb{C}[A_{\mathbb{C}}^{\text{reg}}]$ of regular functions on $A_{\mathbb{C}}^{\text{reg}}$ is the subalgebra of the quotient field of $\mathbb{C}[A_{\mathbb{C}}]$ generated by $\mathbb{C}[A_{\mathbb{C}}]$ and by $1/(1-e^{-2\alpha})$ for $\alpha \in \Sigma^+$. Its W-invariant elements form the subalgebra $\mathbb{C}[A_{\mathbb{C}}^{\text{reg}}]^W$.

1.1. **The hypergeometric system.** In this subsection we outline the theory of hypergeometric differential equations as developed by Heckman, Opdam and Cherednik. More details and further references can be found in [33] and [34].

Let $\mathrm{S}(\mathfrak{a}_{\mathbb{C}})$ denote the symmetric algebra over $\mathfrak{a}_{\mathbb{C}}$ considered as the space of polynomial functions on $\mathfrak{a}_{\mathbb{C}}^*$, and let $\mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$ be the subalgebra of W-invariant elements. Every $p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})$ defines a constant-coefficient differential operators $\partial(p)$ on $A_{\mathbb{C}}$ and on $\mathfrak{a}_{\mathbb{C}}$ such that $\partial(H)$ is the directional derivative in the direction of H for all $H \in \mathfrak{a}$. The algebra of the differential operators $\partial(p)$ with $p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})$ will also be indicated by $\mathrm{S}(\mathfrak{a}_{\mathbb{C}})$. Let $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}}) := \mathbb{C}[A_{\mathbb{C}}^{\text{reg}}] \otimes \mathrm{S}(\mathfrak{a}_{\mathbb{C}})$ denote the algebra of differential operators on $A_{\mathbb{C}}$ with coefficients in $\mathbb{C}[A_{\mathbb{C}}^{\text{reg}}]$. The Weyl group W acts on $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}})$ according to

$$w\big(\varphi \otimes \partial(p)\big) := w\varphi \otimes \partial(wp).$$

We write $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}})^W$ for the subspace of W-invariant elements. The space $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}}) \otimes \mathbb{C}[W]$ can be endowed with the structure of an associative algebra with respect to the product

$$(D_1 \otimes w_1)\cdot(D_2 \otimes w_2) = D_1 w_1(D_2) \otimes w_1 w_2,$$

where the action of W on differential operators is defined by $(wD)(wf) := w(Df)$ for every sufficiently differentiable function f. It is also a left $\mathbb{C}[A_{\mathbb{C}}^{\text{reg}}]$-module. Considering $D \in \mathbb{D}(A_{\mathbb{C}}^{\text{reg}})$ as element of $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}}) \otimes \mathbb{C}[W]$, we shall usually write D instead of $D \otimes 1$. The elements of the algebra $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}}) \otimes \mathbb{C}[W]$ are called the differential-reflection operators on $A_{\mathbb{C}}^{\text{reg}}$. The differential-reflection operators act on functions f on $A_{\mathbb{C}}^{\text{reg}}$ according to $(D \otimes w)f := D(wf)$.

Define a linear map $\Upsilon : \mathbb{D}(A_{\mathbb{C}}^{\text{reg}}) \otimes \mathbb{C}[W] \to \mathbb{D}(A_{\mathbb{C}}^{\text{reg}})$ by

$$\Upsilon\Big(\sum_j D_j \otimes w_j\Big) := \sum_j D_j.$$

Then $\Upsilon(Q)f = Qf$ for all $Q \in \mathbb{D}(A_{\mathbb{C}}^{\text{reg}}) \otimes \mathbb{C}[W]$ and all W-invariant f on $A_{\mathbb{C}}^{\text{reg}}$. Moreover, Υ establishes an algebra homomorphism of

$$(\mathbb{D}(A_{\mathbb{C}}^{\text{reg}}) \otimes \mathbb{C}[W])^{1\otimes\mathbb{C}[W]} := \{Q \in \mathbb{D}(A_{\mathbb{C}}^{\text{reg}}) \otimes \mathbb{C}[W] : Q\cdot(1\otimes w) = (1\otimes w)\cdot Q \text{ for all } w \in W\}$$

into $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}})^W$ (see [14], Lemma 1.2.2).

Definition 1.1. ([5]) Let $m \in \mathcal{M}$ and $H \in \mathfrak{a}_{\mathbb{C}}$. The *Dunkl-Cherednik operator* $T(H,m) \in \mathbb{D}(A_{\mathbb{C}}^{\text{reg}}) \otimes \mathbb{C}[W]$ is defined by

$$T(H,m) := \partial(H) - \rho(m)(H) + \sum_{\alpha\in\Sigma^+} m_\alpha \alpha(H)(1-e^{-2\alpha})^{-1} \otimes (1-r_\alpha)$$

where

$$\rho(m) := \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha \in \mathfrak{a}_{\mathbb{C}}^*. \tag{5}$$

The Dunkl-Cherednik operators map $\mathbb{C}[A_{\mathbb{C}}]$ into itself (see [34], Proposition 2.1). They can also be considered as operators acting on other function spaces, for instance, on the spaces $C^\infty(A)$ and $C_c^\infty(A)$ of C^∞ and compactly supported C^∞ functions on A, or on the analogous spaces of functions on $\mathfrak{a}$. This is always possible because, as can be seen from the Taylor formula, the term $1-r_\alpha$ cancels the apparent singularity on A and $\mathfrak{a}$ arising from the denominator $1-e^{-2\alpha}$.

A remarkable property of the Dunkl-Cherednik operators is that they commute (cf. [33], Section 2). Therefore the map $H \mapsto T(H,m)$ on $\mathfrak{a}_{\mathbb{C}}$ extends uniquely to an algebra homomorphism of $\mathrm{S}(\mathfrak{a}_{\mathbb{C}})$ into $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}}) \otimes \mathbb{C}[W]$. For $p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})$ we set $D(p,m) := \Upsilon\big(T(p,m)\big)$. If $p \in \mathrm{S}(\mathfrak{a}_{\mathbb{C}})^W$, then

$$T(p,m) \in (\mathbb{D}(A_{\mathbb{C}}^{\text{reg}}) \otimes \mathbb{C}[W])^{1\otimes\mathbb{C}[W]}$$

(see [33], Theorem 2.12(2)), and hence $D(p,m) \in \mathbb{D}(A_{\mathbb{C}}^{\text{reg}})^W$.

Let $p_L \in \mathrm{S}(\mathfrak{a}_\mathbb{C})^W$ be the polynomial defined by $p_L(\lambda) := \langle \lambda, \lambda \rangle$ for $\lambda \in \mathfrak{a}_\mathbb{C}^*$. Then

$$D(p_L, m) = L(m) + \langle \rho(m), \rho(m) \rangle,$$

where

$$L(m) := L_A + \sum_{\alpha \in \Sigma^+} m_\alpha \coth \alpha \, \partial_\alpha \tag{6}$$

and L_A is the Laplace operator on A (see [16], Theorem 2.2). In (6) we have set for all $\alpha \in \Sigma^+$

$$\partial_\alpha := \partial(A_\alpha) = \frac{1}{2} \langle \alpha, \alpha \rangle \partial(H_\alpha)$$

and

$$\coth \alpha := \frac{1 + e^{-2\alpha}}{1 - e^{-2\alpha}}.$$

The differential operator $L(m)$ generalizes to arbitrary multiplicity functions m the radial component of the Laplace operator on a Riemannian symmetric space G/K of noncompact type with respect to the left action of K.

The algebra

$$\mathbb{D}(\mathfrak{a}, \Sigma, m) := \{D(p, m) : p \in \mathrm{S}(\mathfrak{a}_\mathbb{C})^W\}$$

is a commutative subalgebra of $\mathbb{D}(A_\mathbb{C}^{\mathrm{reg}})^W$ coinciding with the commutant $\{Q \in \mathbb{D}(A_\mathbb{C}^{\mathrm{reg}})^W : L(m)Q = QL(m)\}$ of $L(m)$ in $\mathbb{D}(A_\mathbb{C}^{\mathrm{reg}})^W$. It is called the algebra of hypergeometric differential operators associated with the data $(\mathfrak{a}, \Sigma, m)$. It is the analogue, for arbitrary multiplicity functions, of the commutative algebra of the radial parts on A of the invariant differential operators on a Riemannian symmetric space of noncompact type.

The map $\gamma(m) : \mathbb{D}(\mathfrak{a}, \Sigma, m) \to \mathrm{S}(\mathfrak{a}_\mathbb{C})^W$ defined by

$$\gamma(m)\big(D(p, m)\big)(\lambda) := p(\lambda) \tag{7}$$

is called the Harish-Chandra homomorphism. It defines an algebra isomorphism of $\mathbb{D}(\mathfrak{a}, \Sigma, m)$ onto $\mathrm{S}(\mathfrak{a}_\mathbb{C})^W$ (see [14], Theorem 1.3.12 and Remark 1.3.14). From Chevalley's theorem it therefore follows that $\mathbb{D}(\mathfrak{a}, \Sigma, m)$ is generated by $l(= \dim \mathfrak{a})$ elements.

Let $\lambda \in \mathfrak{a}_\mathbb{C}^*$ be fixed. The system of differential equations

$$D(p, m)\varphi = p(\lambda)\varphi, \qquad p \in \mathrm{S}(\mathfrak{a}_\mathbb{C})^W, \tag{8}$$

is called the hypergeometric system of differential equations with spectral parameter λ associated with the data $(\mathfrak{a}, \Sigma, m)$. The differential equation corresponding to the polynomial p_L is

$$L(m)\varphi = \big(\langle \lambda, \lambda \rangle - \langle \rho(m), \rho(m) \rangle\big)\varphi. \tag{9}$$

For geometric multiplicities, the hypergeometric system (8) agrees with the system of differential equations on A defining Harish-Chandra's spherical function of spectral parameter λ.

Example 1.2 (The Euclidean case)**.** When $m \equiv 0$, the root system Σ plays no role. $\mathbb{D}(\mathfrak{a}) := \mathbb{D}(\mathfrak{a}, \Sigma, 0)$ consists of the differential operators with constant coefficients. Indeed $T(p, 0) = D(p, 0) = \partial(p)$ for all $p \in \mathrm{S}(\mathfrak{a}_\mathbb{C})^W$. The operator $L(0)$ is the Laplacian L_A.

Example 1.3 (The rank-one case)**.** The rank-one case corresponds to triples $(\mathfrak{a}, \Sigma, m)$ in which $\mathfrak{a}$ is one dimensional. Then the set Σ^+ consists at most of two elements: α and, possibly, 2α. By setting $H_\alpha/2 \equiv 1$ and $\alpha \equiv 1$, we identify $\mathfrak{a}$ and $\mathfrak{a}^*$ with $\mathbb{R}$, and their complexifications $\mathfrak{a}_\mathbb{C}$ and $\mathfrak{a}_\mathbb{C}^*$ with $\mathbb{C}$. The Weyl chamber $\mathfrak{a}^+$ coincides with the half-line $(0, +\infty)$. The Weyl group W reduces to $\{-1, 1\}$ acting on $\mathbb{R}$ and $\mathbb{C}$ by multiplication. Moreover $\rho(m) \equiv m_\alpha/2 + m_{2\alpha}$. We normalize the inner product so that $\langle \alpha, \alpha \rangle = 1$. The algebra $\mathrm{S}(\mathfrak{a}_\mathbb{C})^W$ is generated by the polynomial $p_L(\lambda) = \lambda^2$. Hence $D(p_L, m) = L(m) + \rho^2$ generates $\mathbb{D}(\mathfrak{a}, \Sigma, m)$. The hypergeometric differential system with spectral parameter $\lambda \in \mathbb{C}$ is equivalent to the single Jacobi differential equation

$$\frac{d^2\varphi}{dz^2} + \big(m_\alpha \coth z + m_{2\alpha} \coth(2z)\big) \frac{d\varphi}{dz} = (\lambda^2 - \rho^2)\varphi. \tag{10}$$

The function $z \mapsto e^z$ maps $\mathfrak{a}_\mathbb{C} \equiv \mathbb{C}$ onto $A_\mathbb{C} \equiv \mathbb{C}^\times$. Hence $A_\mathbb{C}^{\mathrm{reg}} \equiv \mathbb{C}\setminus\{0, \pm 1\}$. The change of variable $\zeta := (1 - \cosh z)/2$ transforms (10) into the hypergeometric differential equation

$$\zeta(1-\zeta)\frac{d^2\psi}{d\zeta^2} + [c - (1+a+b)\zeta]\frac{d\psi}{d\zeta} - ab\,\zeta = 0$$

with parameters

$$a = \frac{\lambda+\rho}{2}, \qquad b = \frac{-\lambda+\rho}{2}, \qquad c = \frac{m_\alpha + m_{2\alpha}+1}{2}.$$

At every point $h \in A_\mathbb{C}^{\mathrm{reg}}$, the space $\mathcal{S}(m;\lambda)(h)$ of local solutions of (8) near h consists of holomorphic functions and its dimension is the order $|W|$ of the Weyl group (cf. [14], Corollary 4.1.8. See also [34], Theorem 6.7). Thus the hypergeometric system is holonomic with holonomic rank $|W|$. For "generic" $\lambda \in \mathfrak{a}_\mathbb{C}^*$, a basis for the solution space of (8) on A^+ is constructed by means of the so-called Harish-Chandra series.

1.2. **The Harish-Chandra series.** As in the classical theory of spherical functions on Riemannian symmetric spaces, the explicit expression of the differential equation (9) suggested Heckman and Opdam [20] to look for solutions on A^+ of the hypergeometric system (8) with spectral parameter λ which are of the form

$$\Phi(m;\lambda,a) = e^{(\lambda-\rho)(\log a)} \sum_{\mu\in\Lambda} \Gamma_\mu(m;\lambda) e^{-\mu(\log a)}, \qquad a \in A^+.$$

Here $\Lambda := \left\{\sum_{j=1}^l n_j\alpha_j : n_j \in \mathbb{N}_0\right\}$ is the positive semigroup generated by the fundamental system of simple roots $\Pi := \{\alpha_1, \dots, \alpha_l\}$ in Σ^+. For $\mu \in \Lambda \setminus \{0\}$, the coefficients $\Gamma_\mu(m;\lambda)$ are rational functions of $\lambda \in \mathfrak{a}_\mathbb{C}^*$ determined from the recursion relations

$$\langle \mu, \mu - 2\lambda\rangle \Gamma_\mu(m;\lambda) = 2\sum_{\alpha\in\Sigma^+} m_\alpha \sum_{\substack{k\in\mathbb{N}\\ \mu-2k\alpha\in\Lambda}} \Gamma_{\mu-2k\alpha}(m;\lambda)\langle \mu+\rho-2k\alpha-\lambda, \alpha\rangle,$$

with initial condition $\Gamma_0(m;\lambda) = 1$, which are derived by formally inserting the series for Φ into the differential equation (9). Let $\ell(\mu) := \sum_{j=1}^l n_j$ denote the level of $\mu = \sum_{j=1}^l n_j\alpha_j \in \Lambda$. It is easy to check by induction on $\ell(\mu)$ that the recurrence relations imply $\Gamma_\mu(m;\lambda) = 0$ unless $\mu = \sum_{j=1}^l n_j\alpha_j$ with $n_j \geq 0$ and n_j even for all $j = 1, \dots, l$. Hence the function $\Phi(m;\lambda,a)$ is in fact a sum over 2Λ, that is

$$\Phi(m;\lambda,a) = e^{(\lambda-\rho)(\log a)} \sum_{\mu\in2\Lambda} \Gamma_\mu(m;\lambda) e^{-\mu(\log a)}, \qquad a \in A^+. \tag{11}$$

The relations yield unique solutions $\Gamma_\mu(m;\lambda)$ provided $\langle \mu, \mu-2\lambda\rangle \neq 0$ for all $\mu \in 2\Lambda$.

The function $\Phi(m;\lambda,a)$ is called the Harish-Chandra series. For λ in the set

$$\mathfrak{a}_{\mathbb{C},\Lambda}^* := \{\lambda \in \mathfrak{a}_\mathbb{C}^* : \langle \mu, \mu-2\lambda\rangle \neq 0 \text{ for all } \mu \in 2\Lambda\}, \tag{12}$$

it is a well-defined formal series that converges absolutely and uniformly on compact subsets of A^+. In fact, the Harish-Chandra series extends as a holomorphic function of λ on a much larger subset of $\mathfrak{a}_\mathbb{C}^*$.

Theorem 1.4. ([32], Corollary 2.3) *There is a connected and simply connected open subset U of T containing the identity element e such that the Harish-Chandra series $\Phi(m;\lambda,h)$ extends as a meromorphic function of $(m,\lambda,h) \in \mathcal{M} \times \mathfrak{a}_\mathbb{C}^* \times A^+U$. Its singularities are at most simple poles located along the hyperplanes of the form $\mathcal{M} \times \mathcal{H}_{n,\alpha} \times A^+U$, where*

$$\mathcal{H}_{n,\alpha} := \{\lambda \in \mathfrak{a}_\mathbb{C}^* : \lambda_\alpha = n\} \tag{13}$$

is a complex hyperplane in $\mathfrak{a}_\mathbb{C}^*$ corresponding to some $\alpha \in \Sigma_{\mathrm{i}}^+$ and $n \in \mathbb{N}$. □

The set U in Theorem 1.4 is chosen so that the function log is single valued on it. Then all functions $e^{(\lambda-\rho)(\log h)}$ ($\lambda \in \mathfrak{a}_{\mathbb{C}}^*$) are single valued and holomorphic on A^+U.

By construction, the Harish-Chandra series are solutions of the differential equation (9). As in the Riemannian case, one wishes to use them series to build a basis for solution space of the entire hypergeometric system with spectral parameter λ. This is possible when $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is "generic".

Definition 1.5. We say that $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is *generic* if $\lambda_\alpha \notin \mathbb{Z}$ for all $\alpha \in \Sigma$.

Observe that the set of generic elements in $\mathfrak{a}_{\mathbb{C}}^*$ is W-invariant. Indeed, let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ be generic. Then for all $\alpha, \beta \in \Sigma$ we have $(r_\beta\lambda)_\alpha = \lambda_{r_\beta\alpha} \notin \mathbb{Z}$.

Corollary 1.6. (see [14], Corollary 4.2.6) *Let U be the open subset of T from Theorem 1.4. If $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is generic, then the set $\{\Phi(m; w\lambda, a) : w \in W\}$ is a basis of the solution space on A^+U of the hypergeometric system (8) with spectral parameter λ.* □

Example 1.7 (The Euclidean case)**.** In this case we have plainly $\Phi(0; \lambda, h) = e^{\lambda(\log h)}$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^* \equiv \mathbb{C}^l$.

Example 1.8 (The rank-one case)**.** The solution of the Jacobi differential equation (10) on $(0, +\infty)$ that behaves asymptotically as $e^{(\lambda-\rho)t}$ for $t \to +\infty$ is

$$\Phi(m; \lambda, t) = (2\sinh t)^{\lambda-\rho}\, {}_2F_1\left(\frac{\rho-\lambda}{2}, \frac{-m_\alpha/2+1-\lambda}{2}; 1-\lambda; -\sinh^{-2} t\right),$$

where ${}_2F_1$ denotes the Gaussian hypergeometric function. The function $\Phi(m; \lambda, t)$ coincides with the Jacobi function of second kind $\Phi_\nu^{(a,b)}(t)$ with parameters $a = (m_\alpha + m_{2\alpha} - 1)/2$, $b = (m_{2\alpha} - 1)/2$ and $\nu = -i\lambda$ (see [22], Section 2).

Example 1.9 (The complex case)**.** Let m be geometric multiplicity of a reduced system Σ. If $m_\alpha = 2$ for all $\alpha \in \Sigma$, then m corresponds to a Riemannian symmetric space of the noncompact type G/K with G complex. The triple $(\mathfrak{a}, \Sigma, m)$ will be said to correspond to a complex case. The constant multiplicity function with all values equal to $k \in \mathbb{C}$ on Σ will be indicated by the symbol k. In the complex case, we have

$$\Phi(2; \lambda, a) = \Delta(a)^{-1} e^{\lambda(\log a)}. \tag{14}$$

where

$$\Delta := \prod_{\alpha \in \Sigma^+} (e^\alpha - e^{-\alpha}) \tag{15}$$

is the Weyl denominator.

2. c-Functions

Let $(\mathfrak{a}, \Sigma, m)$ be a triple as in Section 1. Let Θ denote an arbitrary subset of the fundamental system $\Pi = \{\alpha_1, \dots, \alpha_l\}$ associated with a choice Σ^+ of positive roots in Σ. The set $\langle\Theta\rangle$ of elements in Σ which can be written as linear combinations of elements from Θ is a subsystem of Σ. Its Weyl group W_Θ is generated by the reflections $r_j := r_{\alpha_j}$ with $\alpha_j \in \Theta$. We denote by $\langle\Theta\rangle_{\mathrm{i}}$ and $\langle\Theta\rangle_{\mathrm{u}}$ respectively the indivisible and unmultipliable roots in $\langle\Theta\rangle$. Their subset of positive roots are indicated with $\langle\Theta\rangle_{\mathrm{i}}^+$ and $\langle\Theta\rangle_{\mathrm{u}}^+$. Then

$$W_\Theta(\Sigma^+ \setminus \langle\Theta\rangle^+) \subset \Sigma^+ \setminus \langle\Theta\rangle^+ \qquad \text{and} \qquad W_\Theta(\Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+) \subset \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+. \tag{16}$$

Definition 2.1. For a multiplicity function m on Σ, a root $\alpha \in \Sigma^+$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we set

$$c_\alpha^+(m; \lambda) := \frac{\Gamma\Big(\lambda_\alpha + \frac{m_{\alpha/2}}{4}\Big)}{\Gamma\Big(\lambda_\alpha + \frac{m_{\alpha/2}}{4} + \frac{m_\alpha}{2}\Big)},$$

$$c_\alpha^-(m; \lambda) := \frac{\Gamma\Big(-\lambda_\alpha - \frac{m_{\alpha/2}}{4} - \frac{m_\alpha}{2} + 1\Big)}{\Gamma\Big(-\lambda_\alpha - \frac{m_{\alpha/2}}{4} + 1\Big)}.$$

Let $\Theta \subset \Pi$ be any subset. We define the functions $c_\Theta^\pm$ and $c_\Theta^{\pm,c}$ by

$$c_\Theta^+(m;\lambda) := \prod_{\alpha\in\langle\Theta\rangle^+} c_\alpha^+(m;\lambda),$$
$$c_\Theta^{+,c}(m;\lambda) := \prod_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} c_\alpha^+(m;\lambda),$$
$$c_\Theta^-(m;\lambda) := \prod_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} c_\alpha^-(m;\lambda),$$
$$c_\Theta^{-,c}(m;\lambda) := \prod_{\alpha\in\langle\Theta\rangle^+} c_\alpha^-(m;\lambda),$$

with the conventions

$$c_\emptyset^+ \equiv c_\Pi^{+,c} := 1 \qquad \text{and} \qquad c_\Pi^- \equiv c_\emptyset^{-,c} := 1.$$

Remark 2.2. (a) The interchange of $\langle\Theta\rangle^+$ and $\Sigma^+ \setminus \langle\Theta\rangle^+$ in the formulas for c_Θ^+ and c_Θ^- is motivated by the definition of the Θ-hypergeometric functions in Section 3.

(b) The W-invariance of m and (16) imply that $c_\Theta^-(m;\lambda)$ and $c_\Theta^{+,c}(m;\lambda)$ are W_Θ-invariant functions of $\lambda \in \mathfrak{a}_\mathbb{C}^*$.

(c) The duplication formula for the gamma function

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2) \tag{17}$$

yields

$$c_\Theta^+(m;\lambda) = \prod_{\alpha\in\langle\Theta\rangle_{\mathrm{i}}^+} \sqrt{\pi}\,\frac{2^{-(\lambda_\alpha+m_\alpha/2)+1}\,\Gamma(\lambda_\alpha)}{\Gamma\Big(\frac{\lambda_\alpha}{2}+\frac{m_\alpha}{4}+\frac{1}{2}\Big)\Gamma\Big(\frac{\lambda_\alpha}{2}+\frac{m_\alpha}{4}+\frac{m_{2\alpha}}{2}\Big)} \tag{18}$$

$$c_\Theta^-(m;\lambda) = \prod_{\alpha\in\Sigma_{\mathrm{i}}^+\setminus\langle\Theta\rangle_{\mathrm{i}}^+} \frac{1}{\sqrt{\pi}}\,\frac{\Gamma\Big(-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}+\frac{1}{2}\Big)\Gamma\Big(-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}-\frac{m_{2\alpha}}{2}+1\Big)}{2^{\lambda_\alpha+m_\alpha/2}\,\Gamma(-\lambda_\alpha+1)}. \tag{19}$$

Definition 2.1 gives a concise representation of the functions $c_\Theta^\pm$. Nevertheless in formulas (18) and (19) the location of possible singularities and zeros of these functions is more intelligible. See Lemma 2.7 below.

(d) When $\alpha/2$ is not a root (e.g. when Σ is reduced), then

$$c_\alpha^+(m;\lambda) := \frac{\Gamma(\lambda_\alpha)}{\Gamma(\lambda_\alpha+m_\alpha/2)}$$
$$c_\alpha^-(m;\lambda) := \frac{\Gamma(-\lambda_\alpha-m_\alpha/2+1)}{\Gamma(-\lambda_\alpha+1)}.$$

Therefore, when Σ is reduced and m_α is even for all α,

$$c_\Theta^+(m;\lambda) = \prod_{\alpha\in\langle\Theta\rangle^+}\prod_{k=0}^{m_\alpha/2-1}\frac{1}{\lambda_\alpha+k},$$
$$c_\Theta^-(m;\lambda) = \prod_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+}\prod_{k=-m_\alpha/2+1}^{0}\frac{1}{-\lambda_\alpha+k}$$

are inverse of polynomials.

Example 2.3. (a) The function

$$c(m;\lambda) := \frac{c_\Pi^+(m;\lambda)}{c_\Pi^+(m;\rho(m))} \tag{20}$$

coincides for geometric multiplicities m with the Harish-Chandra c-function for Riemannian symmetric spaces (see e.g. [9], Theorem 4.7.5). Recall that the Harish-Chandra c-function is normalized to have value 1 at $\rho(m)$.

(b) Suppose that m is a geometric multiplicity function corresponding to a NCC symmetric space G/H. Let $\Theta := \Pi_0$ be the fundamental system for the system Σ_0^+ of positive compact roots. Then $\langle\Theta\rangle^+ = \Sigma_0^+$. The functions

$$c_0(m;\lambda) := \frac{c_\Theta^+(m;\lambda)}{c_\Theta^+(m;\rho(m))}$$

$$c_\Omega(m;\lambda) := \frac{c_\Theta^-(m;\lambda)}{c_\Theta^-(m;\rho(m))}$$

are respectively the Harish-Chandra c-function for the root system Σ_0 and the function c_Ω of [24].

Lemma 2.4. *Let $m \in \mathcal{M}$ be fixed. Then for all $\alpha \in \Sigma^+$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$*

$$c_\alpha^+(m;\lambda) = \frac{\sin\Big[\pi\Big(\lambda_\alpha + \frac{m_{\alpha/2}}{4} + \frac{m_\alpha}{2}\Big)\Big]}{\sin\Big[\pi\Big(\lambda_\alpha + \frac{m_{\alpha/2}}{4}\Big)\Big]}\, c_\alpha^-(m;\lambda).$$

Proof. Immediate consequence of the formula $\Gamma(z) = \frac{\pi}{\sin(\pi z)}\frac{1}{\Gamma(1-z)}$. □

To study the singularities of the functions $c_\Theta^\pm$ introduced in this section, it is convenient to introduce separate notation for the numerator and the denominator of the function c_Θ^- according to formula (19).

Definition 2.5. The functions

$$n_\Theta^-(m;\lambda) := \prod_{\alpha\in\Sigma_\mathrm{i}^+\setminus\langle\Theta\rangle_\mathrm{i}^+} \Gamma\Big(-\frac{\lambda_\alpha}{2} - \frac{m_\alpha}{4} + \frac{1}{2}\Big)\Gamma\Big(-\frac{\lambda_\alpha}{2} - \frac{m_\alpha}{4} - \frac{m_{2\alpha}}{2} + 1\Big) \tag{21}$$

$$d_\Theta^-(m;\lambda) := \prod_{\alpha\in\Sigma_\mathrm{i}^+\setminus\langle\Theta\rangle_\mathrm{i}^+} 2^{\lambda_\alpha + m_\alpha/2}\sqrt{\pi}\,\Gamma(-\lambda_\alpha+1) \tag{22}$$

are respectively called the *numerator* and the *denominator* of $c_\Theta^-(m;\lambda)$.

Remark 2.6. Suppose $2\alpha \notin \Sigma^+$ for all $\alpha \in \Sigma_\mathrm{i}^+ \setminus \langle\Theta\rangle_\mathrm{i}^+$, which occurs for instance when Σ is reduced. Then we obtain from the duplication formula (17)

$$n_\Theta^-(m;\lambda) := \prod_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} 2^{\lambda_\alpha + m_\alpha/2}\sqrt{\pi}\,\Gamma\Big(-\lambda_\alpha - \frac{m_\alpha}{2} + 1\Big).$$

Recall the notation $\mathcal{H}_{r,\alpha} := \{\lambda \in \mathfrak{a}_\mathbb{C}^* : \lambda_\alpha = r\}$ from (13). Observe that the equality $\mathcal{H}_{n,\alpha} = \mathcal{H}_{m,\beta}$ with $\alpha,\beta \in \Sigma_\mathrm{i}^+$ and $n,m \in \mathbb{Z}$ implies $\alpha = \beta$ and $n = m$. From the singularities of the gamma function we therefore obtain the following lemma.

Lemma 2.7. (a) *The singularities of the meromorphic function $c_\Theta^+(m;\lambda)$ are at most simple poles located along the hyperplanes*

$$\mathcal{H}_{-n,\alpha} \qquad \text{with } \alpha \in \langle\Theta\rangle_\mathrm{i}^+ \text{ and } n \in \mathbb{N}_0.$$

(b) *The singularities of the numerator $n_\Theta^-(m;\lambda)$ are poles located along the hyperplanes*

$$\mathcal{H}_{-\frac{m_\alpha}{2}+(2n-1),\alpha} \qquad \text{with } \alpha \in \Sigma_\mathrm{i}^+ \setminus \langle\Theta\rangle_\mathrm{i}^+ \text{ and } n \in \mathbb{N},$$

$$\mathcal{H}_{-\frac{m_\alpha}{2}-m_{2\alpha}+2n,\alpha} \qquad \text{with } \alpha \in \Sigma_\mathrm{i}^+ \setminus \langle\Theta\rangle_\mathrm{i}^+ \text{ and } n \in \mathbb{N}.$$

Hence all poles are simple if and only if $m_{2\alpha}$ is not an odd integer for all $\alpha \in \Sigma_\mathrm{i}^+ \setminus \langle\Theta\rangle_\mathrm{i}^+$. If $m_{2\alpha}$ is an odd integer for some $\alpha \in \Sigma_\mathrm{i}^+ \setminus \langle\Theta\rangle_\mathrm{i}^+$, then there are at most double poles.

In particular, the poles are always simple when $2\alpha \notin \Sigma^+$ for all $\alpha \in \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+$ (e.g. when Σ is reduced). In this case the singularities of the numerator $n_\Theta^-(m;\lambda)$ are simple poles located along the hyperplanes

$$\mathcal{H}_{-\frac{m_\alpha}{2}+n,\alpha} \qquad \text{with } \alpha \in \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+ \text{ and } n \in \mathbb{N}.$$

(c) *The singularities of the denominator $d_\Theta^-(m;\lambda)$ are simple poles located along the hyperplanes*

$$\mathcal{H}_{n,\alpha} \qquad \text{with } \alpha \in \Sigma_{\mathrm{i}}^+ \setminus \langle\Theta\rangle_{\mathrm{i}}^+ \text{ and } n \in \mathbb{N}.$$

□

Remark 2.8. In the geometric case, it is known that, if 2α is a root, then $m_{2\alpha}$ is odd. It follows in this case by part (b) of Lemma 2.7 that all poles of n_Θ^- are simple if and only if 2α is not a root for all $\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$.

3. Θ-hypergeometric functions

Definition 3.1. Let $\Theta \subset \Pi$ be any subset of simple roots and let $c_\Theta^\pm$ be the functions introduced in Definition 2.1. Let U be the open subset of T from Theorem 1.4. The function on A^+U defined for generic $\lambda \in \mathfrak{a}_\mathbb{C}^*$ by

$$\varphi_\Theta(m;\lambda,h) := c_\Theta^-(m;\lambda) \sum_{w\in W_\Theta} c_\Theta^+(m;w\lambda)\Phi(m;w\lambda,h), \qquad h \in A^+U \tag{23}$$

is called the Θ-*hypergeometric function of spectral parameter* λ. Because of the following Examples 3.3 (a) and (b), the function $\varphi_\Theta(m;\lambda,h)$ will be called the Θ-*spherical function of spectral parameter* λ when the multiplicity function m is geometric.

As a linear combination of the Harish-Chandra series $\Phi(m;w\lambda,h)$, the Θ-hypergeometric function of spectral parameter λ is by construction a solution of the hypergeometric system (8) of spectral parameter λ.

Remark 3.2. The Θ-hypergeometric functions appeared first in the appendix of the paper [28] of G. Ólafsson and the present author, where they have been employed to prove a conjecture on the location of the singularities in the spectral parameter λ of the spherical functions on NCC symmetric spaces.

Example 3.3. (a) When $\Theta = \Pi$, then

$$\varphi_\Pi(m;\lambda,h) = \sum_{w\in W} c_\Pi^+(m;w\lambda)\Phi(m;w\lambda,h)$$

has been first considered by Heckman and Opdam in [17]. The hypergeometric function associated with the root system Σ is

$$F(m;\lambda,h) := \frac{\varphi_\Pi(m;\lambda,h)}{c_\Pi^+(m;\rho(m))} = \sum_{w\in W} c(m;w\lambda)\Phi(m;w\lambda,h),$$

where $c(m;\lambda)$ is Harish-Chandra's c-function of formula (20). The chosen normalization for $F(m;\lambda,h)$ is due to the requirement $F(m;\lambda,e)=1$, in analogy to the classical requirement imposed on the spherical functions on Riemannian symmetric spaces. The computation of the value at e is a non-trivial matter which has been carried out by Opdam using the so-called shift operators (see e.g. [14], pp. 67–68).

For geometric multiplicities m, the hypergeometric functions $F(m;\lambda,h)$ agrees with Harish-Chandra's spherical functions. In particular, $\varphi_\Pi(m;\lambda,h)$ reduces to a spherical function up to a multiplicative factor depending only on the multiplicities.

(b) Suppose Σ is the restricted root system of a NCC symmetric space (in particular Σ is reduced). Let Π_0 be the fundamental system for the positive compact roots Σ_0^+ and set $\Theta := \Pi_0$. Then $\langle\Theta\rangle = \Sigma_0$ and $W_\Theta = W_0$ is the so-called small Weyl group. Up to a factor depending only on the multiplicities, the Θ-hypergeometric functions reduce to Unterberger's for arbitrary multiplicities, and (because of Ólafsson's expansion [26]) to the spherical functions on NCC symmetric spaces for geometric multiplicities.

(c) When $\Theta = \emptyset$, then $W_\emptyset = \{\mathrm{id}\}$, hence $\varphi_\emptyset(m;\lambda,h) = c_\emptyset^-(m;\lambda)\Phi(m;\lambda,h)$.

The function φ_Θ is a priori defined only for $\lambda \in \mathfrak{a}_\mathbb{C}^*$ generic and $h \in A^+U$. In the case $\Theta = \Pi$, Heckman and Opdam proved that $\varphi_\Pi(m;\lambda,h)$ extends as a W-invariant holomorphic function in $\mathcal{M} \times \mathfrak{a}_\mathbb{C}^* \times U_\Pi$, where U_Π denotes a W-invariant tubular neighborhood of A in $A_\mathbb{C}$ (see e.g. [14], §4.3–4.4). Their result parallels the regularity properties of the spherical functions on Riemannian symmetric spaces. For arbitrary $\Theta \subset \Pi$, there exists a certain W_Θ-invariant open domain $U_\Theta \subset A_\mathbb{C}$ such that the right-hand side of (23) extends as a W_Θ-invariant holomorphic function of $h \in U_\Theta$, as a W_Θ-invariant meromorphic function of $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and as entire function of $m \in \mathcal{M}$. See Theorem 3.5 below. The domain U_Θ is a tubular neighborhood in $A_\mathbb{C}$ of an open domain A_Θ constructed as follows. Let $\overline{\mathfrak{a}^+}$ denote the closure of the positive Weyl chamber, and let $\overline{A^+} := \exp(\overline{\mathfrak{a}^+})$. We define

$$\mathfrak{a}_\Theta := W_\Theta(\overline{\mathfrak{a}^+})^0 \quad \text{and} \quad A_\Theta := \exp \mathfrak{a}_\Theta = W_\Theta(\overline{A^+})^0, \tag{24}$$

where ^{0} denotes the interior. The set $\overline{\mathfrak{a}_\Theta} = W_\Theta(\overline{\mathfrak{a}^+})$ is the smallest cone in $\mathfrak{a}$ which is closed, W_Θ-invariant and contains $\overline{\mathfrak{a}^+}$. It is a union of closed Weyl chambers. The open polyhedral cone in $\mathfrak{a}^*$

$$C_\Theta := \sum_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \mathbb{R}^+ H_\alpha$$

has dual cone

$$\begin{aligned} C_\Theta^* :&= \{H \in \mathfrak{a} : \langle H, Y\rangle \geq 0 \text{ for all } Y \in C_\Theta\} \\ &= \{H \in \mathfrak{a} : \alpha(H) \geq 0 \text{ for all } \alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+\}. \end{aligned}$$

As intersection of the closed hyperplanes defined by roots, C_Θ^* is a convex cone which is a union of closed Weyl chambers in $\mathfrak{a}$. The geometric properties of A_Θ are given by the following lemma.

Lemma 3.4. *We have*

$$\overline{\mathfrak{a}_\Theta} = C_\Theta^*.$$

Consequently, $\overline{\mathfrak{a}_\Theta}$ is a closed W_Θ-invariant convex cone in $\mathfrak{a}$ and also its interior $\mathfrak{a}_\Theta$ is convex.

Proof. The cone C_Θ^* is W_Θ-invariant. Indeed for $H \in C_\Theta^*$, $\alpha_j \in \Theta$ and $\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$ one has $\alpha(r_j(H)) = (r_j\alpha)(H) \geq 0$ because $r_j\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$ by (16). Hence $r_j(H) \in C_\Theta^*$. By definition, $\overline{\mathfrak{a}^+} \subset C_\Theta^*$. Thus $\overline{\mathfrak{a}_\Theta} \subset C_\Theta^*$.

Suppose now $\overline{\mathfrak{a}_\Theta} \neq C_\Theta^*$. Both cones are unions of closed Weyl chambers. Moreover C_Θ^* is convex. Hence there is a closed Weyl chamber $C \subset C_\Theta^* \setminus \mathfrak{a}_\Theta$ such that $\overline{C} \cap \overline{\mathfrak{a}_\Theta}$ is the wall F of a chamber in $\overline{\mathfrak{a}_\Theta}$. Let α be the element in Σ^+ such that $\alpha(H) = 0$ for all $H \in F$, and let C' be the Weyl chamber of $\overline{\mathfrak{a}_\Theta}$ with wall F. If $\alpha \in \langle\Theta\rangle^+$, then $C = r_\alpha(C') \subset W_\Theta(\overline{\mathfrak{a}^+}) = \overline{\mathfrak{a}_\Theta}$. Thus $\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$. Now, let $H \in C^0$. Then $r_\alpha(H) \in W_\Theta(\mathfrak{a}^+)$. Hence there is $H' \in \mathfrak{a}^+$ and $w_0 \in W_\Theta$ such that $r_\alpha(H) = w_0(H')$. Since $w_0^{-1}\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$, we have

$$\alpha(r_\alpha(H)) = \alpha(w_0(H')) = (w_0^{-1}\alpha)(H') > 0.$$

Since H and $r_\alpha(H)$ lie on opposite sides of the hyperplane $\{H \in \mathfrak{a} : \alpha(H) = 0\}$, it follows that $\alpha(H) < 0$ and $H \notin C_\Theta^*$, against the choice of C. This contradiction thus implies that $\overline{\mathfrak{a}_\Theta} = C_\Theta^*$. $\square$

The regularity properties of the Θ-hypergeometric functions are given by the following theorem. Its proof depends heavily on the study of the hypergeometric differential equations of Heckman and Opdam. Since it essentially appeared already in [28], we only present here an outline of its proof.

Theorem 3.5. *There exists a W_Θ-invariant tubular neighborhood U_Θ of A_Θ in $A_\mathbb{C}$ such that the function*

$$\frac{\varphi_\Theta(m;\lambda,h)}{n_\Theta^-(m;\lambda)} = \frac{1}{d_\Theta^-(m;\lambda)} \sum_{w\in W_\Theta} c_\Theta^+(m;w\lambda)\Phi(m;w\lambda;h)$$

extends as a holomorphic function of $(m,\lambda,h)\in\mathcal{M}\times\mathfrak{a}_\mathbb{C}^*\times U_\Theta$ and satisfies

$$\frac{\varphi_\Theta(m;w\lambda,h)}{n_\Theta^-(m;w\lambda)} = \frac{\varphi_\Theta(m;\lambda,h)}{n_\Theta^-(m;\lambda)} = \frac{\varphi_\Theta(m;\lambda,wh)}{n_\Theta^-(m;\lambda)}$$

for all $w\in W_\Theta$ and $(m,\lambda,h)\in\mathcal{M}\times\mathfrak{a}_\mathbb{C}^*\times U_\Theta$.

In particular, the λ-singularities of the Θ-hypergeometric function $\varphi_\Theta(m;\lambda,h)$ are contained (counted with multiplicities) in the polar set of the numerator function $n_\Theta^-(m;\lambda)$. Furthermore, the λ-singularities are at most simple poles when Σ is a reduced root system.

Proof. (cf. [28], Theorem 8.7) The case $\Theta=\emptyset$ follows from Theorem 1.4. We can therefore assume $\Theta\neq\emptyset$. Let $m\in\mathcal{M}$ and $h\in A^+U$ be fixed. Because of Theorem 1.4 and Lemma 2.7, the possible singularities of $\varphi_\Theta(m;\lambda,h)$ in $\lambda\in\mathfrak{a}_\mathbb{C}^*$ are: (a) poles of $n_\Theta^-(m;\lambda)$; (b) simple poles along the hyperplanes $\mathcal{H}_{n,\alpha}$ with $\alpha\in\langle\Theta\rangle_\mathrm{i}^+$ and $n\in\mathbb{Z}$. We claim that all singularities in (b) are removable. Observe that the intersection of hyperplanes $\mathcal{H}_{n,\alpha}$ for $\alpha\in\langle\Theta\rangle_\mathrm{i}^+$ with hyperplanes $\mathcal{H}_{n,\alpha}$ for $\Sigma_\mathrm{i}^+\setminus\langle\Theta\rangle_\mathrm{i}^+$ are varieties of codimension >1. Hence Hartogs theorem guarantees that the singularities in (b) are removable if and only if the singularities along the same hyperplanes of the function

$$\sum_{w\in W_\Theta} c_\Pi^+(m;w\lambda)\Phi(m;w\lambda,h)$$

are removable. This is the sum considered in Heckman-Opdam's work except that their summation over W has been replaced by the summation over the subgroup W_Θ. The claim follows by applying the method of pole cancellation developed by Opdam in [31]. The extension in the variable $h\in A_\Theta$ is obtained using monodromy arguments. The regularity in $m\in\mathcal{M}$ follows from Theorem 1.4 and Lemma 2.7. □

Corollary 3.6. *Retain the notation of Theorem 3.5 and let $(m,h)\in\mathcal{M}\times U_\Theta$ be fixed. Then*

$$\frac{\varphi_\Theta(m;\lambda,h)}{c_\Theta^-(m;\lambda)}$$

is a holomorphic function of λ on the open set $B_\Theta:=\{\lambda\in\mathfrak{a}_\mathbb{C}^*:\operatorname{Re}\lambda_\alpha<1$ for all $\alpha\in\Sigma_\mathrm{i}^+\setminus\langle\Theta\rangle_\mathrm{i}^+\}$.

□

Example 3.7. Since the root system of a NCC symmetric space G/H is always reduced, the singularities in the spectral parameter λ of the spherical functions on G/H are always simple poles. They are contained in the polar set of the numerator function $n_\Omega(\lambda):=\prod_{\alpha\in\Sigma_+\setminus\Sigma_0^+}\Gamma(-\lambda_\alpha-m_\alpha/2+1)$. In the space parameter a, the spherical functions are defined and real analytic in a tubular neighborhood of $\exp\big(W_0\cdot\overline{\mathfrak{a}^+}\big)^0=\exp(c_{\max}^0)$, where $c_{\max}$ is the maximal cone in $\mathfrak{a}$ associated with the causal structure of G/H.

Example 3.8 (The rank-one case)**.** In the rank-one case, there are only two possibilities for Θ, namely $\Theta=\Pi$ or $\Theta=\emptyset$. The first case yields constant multiples of the Jacobi functions of first kind

$$\varphi_\Pi(m;\lambda,t) = {}_2F_1\left(\frac{m_\alpha+2m_{2\alpha}+2\lambda}{4},\frac{m_\alpha+2m_{2\alpha}-2\lambda}{4};\frac{m_\alpha+2m_{2\alpha}+1}{2};-\sinh^2 t\right).$$

In the case $\Theta=\emptyset$, the function $\varphi_\emptyset$ is a multiple of the Harish-Chandra series (cf. Examples 1.8 and 3.3 (c)). The study of the singular set of the hypergeometric function is very classical (see e.g. [41]). For the geometric case of the NCC symmetric space $\mathrm{SO}_0(1,n)/\mathrm{SO}_0(1,n-1)$ with $n\geq 2$ (corresponding to $m_\alpha=n-1$ and $m_{2\alpha}=0$), the exact list of singularities of $\varphi_\emptyset$ has been quickly determined with the method of Bernstein-Sato polynomials in [28], Corollary 6.4. From the obtained list, one sees that the λ-singularities of $\varphi_\emptyset$ are exactly all (simple) poles of the functions $n_\emptyset^-$ when m_α is odd. When m_α is even, there are only finitely many poles (and they are all simple). See also the following example.

Example 3.9 (The even multiplicity case)**.** A multiplicity function m on a reduced root system Σ is said to be *(positive) even* if $m_\alpha \in 2\mathbb{N}$ or all $\alpha \in \Sigma$. In the geometric case, even multiplicities correspond to Riemannian spaces G/K with the property that all Cartan subalgebras in the Lie algebra $\mathfrak{g}$ of G are conjugate under the adjoint group of $\mathfrak{g}$. The easiest examples correspond to the cases in which G is complex, where all multiplicities are equal to 2. In the even multiplicity case the result in Theorem 3.5 can be sharpened by showing that the λ-singularities of the Θ-hypergeometric functions are located on a specific *finite* union of complex affine hyperplanes. Indeed, let $\Theta \subset \Pi$ and define

$$e_\Theta^-(m;\lambda) := \prod_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \prod_{k=-m_\alpha/2+1}^{m_\alpha/2-1} (\lambda_\alpha - k). \tag{25}$$

for $\lambda \in \mathfrak{a}_\mathbb{C}^*$. Then there is a W_Θ-invariant tubular neighborhood U_Θ in $A_\mathbb{C}$ of A_Θ such that the function

$$e_\Theta^-(m;\lambda)\,\varphi_\Theta(m;\lambda,h)$$

extends as a W_Θ-invariant holomorphic function of $(\lambda,h) \in \mathfrak{a}_\mathbb{C}^* \times U_\Theta$.

In the complex case an explicit formula for the Θ-hypergeometric functions $\varphi_\Theta(2;\lambda,h)$ is moreover available. According to Remark 2.2 (a), we have

$$c_\Theta^+(2;\lambda) = \Big[\prod_{\alpha\in\langle\Theta\rangle^+} \lambda_\alpha\Big]^{-1} \qquad \text{and} \qquad c_\Theta^-(2;\lambda) = \Big[\prod_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} (-\lambda_\alpha)\Big]^{-1}.$$

Consider the factorization $\Delta = \Delta_\Theta^+\Delta_\Theta^-$ of the Weyl denominator (15) with

$$\Delta_\Theta^+ := \prod_{\alpha\in\langle\Theta\rangle^+} (e^\alpha - e^{-\alpha}) \qquad \text{and} \qquad \Delta_\Theta^- := \prod_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} (e^\alpha - e^{-\alpha}).$$

Then there is a W_Θ-invariant tubular neighborhood U_Θ of A_Θ in $A_\mathbb{C}$ such that for all $(\lambda,h) \in \mathfrak{a}_\mathbb{C}^* \times U_\Theta$

$$\varphi_\Theta(2;\lambda,h) = \frac{c_\Theta^-(2;\lambda)}{\Delta_\Theta^-(h)}\,\varphi_\Theta^0(2;\lambda,h),$$

where

$$\varphi_\Theta^0(2;\lambda,h) := \frac{c_\Theta^+(2;\lambda)}{\Delta_\Theta^+(h)} \sum_{w\in W_\Theta} \det(w)\, e^{w\lambda(\log h)}$$

is the Θ-hypergeometric function with $m_\alpha = 2$ for all $\alpha \in \langle\Theta\rangle$ and $m_\alpha = 0$ for $\alpha \in \Sigma^+\setminus\langle\Theta\rangle^+$. In the case $\Theta = \Pi$, we have $\varphi_\Theta(2;\lambda,h) = \varphi_\Theta^0(2;\lambda,h)$, and the above formula reduces to Harish-Chandra's explicit formulas for the spherical functions on G/K with G complex. See [12], Section 14.

We refer to [30] for more information on the even multiplicity case, in particular for the classification of the NCC symmetric spaces with even multiplicities, and for the proof of the above statements.

The definition of the Θ-hypergeometric functions, together with analytic continuation, yields the following transition relations linking the Θ-hypergeometric functions for arbitrary Θ to the hypergeometric functions of Heckman and Opdam.

Lemma 3.10. *There is a W_Θ-invariant tubular neighborhood U_Θ of A_Θ in $A_\mathbb{C}$ so that for all $(m,\lambda,h) \in \mathcal{M} \times \mathfrak{a}_\mathbb{C}^* \times U_\Theta$ the following equality of meromorphic functions holds:*

$$\varphi_\Pi(m;\lambda,h) = \sum_{w\in W_\Theta\setminus W} \frac{c_\Theta^{+,c}(m;w\lambda)}{c_\Theta^-(m;w\lambda)}\,\varphi_\Theta(m;w\lambda,h), \tag{26}$$

where the functions $c_\Theta^{+,c}(m;w\lambda)$ and $c_\Theta^-(m;w\lambda)$ are as in Definition 2.1.

Proof. By W_Θ-invariance in λ of $c_\Theta^{+,c}(m;w\lambda)$, $c_\Theta^-(m;w\lambda)$ and $\varphi_\Theta(m;\lambda,a)$, we obtain for all $m\in\mathcal{M}$, $h\in A^+U$ and for generic $\lambda\in\mathfrak{a}_\mathbb{C}^*$

$$\begin{aligned}\varphi_\Pi(m;\lambda,h) &:= \sum_{w\in W} c_\Pi^+(m;w\lambda)\Phi(m;w\lambda,h)\\ &= \sum_{w\in W_\Theta\backslash W}\sum_{w'\in W_\Theta} c_\Theta^+(m;w'w\lambda)c_\Theta^{+,c}(m;w'w\lambda)\Phi(m;w'w\lambda,h)\\ &= \sum_{w\in W_\Theta\backslash W}\frac{c_\Theta^{+,c}(m;w\lambda)}{c_\Theta^-(m;w\lambda)}\left(c_\Theta^-(m;w\lambda)\sum_{w'\in W_\Theta} c_\Theta^+(m;w'w\lambda)\Phi(m;w'w\lambda,h)\right)\\ &= \sum_{w\in W_\Theta\backslash W}\frac{c_\Theta^{+,c}(m;w\lambda)}{c_\Theta^-(m;w\lambda)}\varphi_\Theta(m;w\lambda,h).\end{aligned}$$

The equality extends by analyticity to $(m,\lambda,h)\in\mathcal{M}\times\mathfrak{a}_\mathbb{C}^*\times U_\Theta$ for a suitable W_Θ-invariant tubular neighborhood U_Θ of A_Θ in $A_\mathbb{C}$. □

Example 3.11. (a) In the geometric case with $\Theta=\emptyset$, equation (26) reduces to the classical Harish-Chandra expansion of the spherical functions on G/K with respect to the Harish-Chandra series $\Phi(m;\lambda,h)$.

(b) When $(\mathfrak{a},\Sigma,m)$ corresponds to a NCC symmetric space G/H and m is geometric, then (26) is Ólafsson's functional relation between spherical functions on G/H and on its Riemannian dual G/K (see [26], Theorem 5.9).

4. Estimates for the Harish-Chandra series

In this section we prove estimates for the Harish-Chandra series $\Phi(m;\lambda,a)$ which are uniform in $a\in A^+$ and locally uniform in $\lambda\in\mathfrak{a}_\mathbb{C}^*$. For the multiplicity function m, we assume $m\in\mathcal{M}^+$ is fixed, and we convene that $m_\alpha=0$ is equivalent to $\alpha\notin\Sigma$.

Our methods are a modification of the renowned estimates of Gangolli [8]. As in [8], we consider the modified Harish-Chandra series

$$\Psi(m;\lambda,a):=\delta\,(m/2;a)\,\Phi(m;\lambda,a), \tag{27}$$

where δ is defined by (2). Suppose $\lambda\in\mathfrak{a}_{\mathbb{C},\Lambda}^*$ (see (12)). The functions Ψ has then a series expansion

$$\Psi(m;\lambda,a)=e^{\lambda(\log a)}\sum_{\mu\in 2\Lambda}a_\mu(m;\lambda)e^{-\mu(\log a)},\qquad a\in A^+, \tag{28}$$

which is obtained as Cauchy product of the series expansions of Φ and $\delta\,(m/2)$. Since

$$|\Psi(m;\lambda,a)|\le e^{\operatorname{Re}\lambda(\log a)}\sum_{\mu\in 2\Lambda}|a_\mu(m;\lambda)|\,e^{-\mu(\log a)},$$

we can obtain estimates for Ψ from those of the coefficients $a_\mu(m;\lambda)$. For this purpose, we first determine a differential equation satisfied by Ψ. Formal insertion of the series expansion for Ψ produces recurrence relations which we can employ inductively for estimating the $a_\mu(m;\lambda)$.

The point of using Ψ instead of Φ is that Ψ satisfies a second order differential equation without first order terms; see equation (31) below. This simplifies the structure of the recurrence relations among the coefficients $a_\mu(m;\lambda)$. Gangolli's estimation procedure for the Harish-Chandra series via the function Ψ has been originally inspired by the classical method of bringing the hypergeometric differential equation in Liouville's normal form (see [8] for further details).

Our final estimates differ from those in [8] in two distinguishing points. First of all, we give more precise information on the behavior of the Harish-Chandra series on the walls of A^+. Secondly, Gangolli's estimates are uniform on the set of all $\lambda\in\mathfrak{a}_\mathbb{C}^*$ satisfying $\operatorname{Re}\langle\lambda,\alpha\rangle\le 0$ for all $\alpha\in\Sigma^+$. This set avoids the singularities of $\Phi(m;\lambda,a)$. Our applications in Section 5 require that we consider the entire $\mathfrak{a}_{\mathbb{C}}^*$. We are not able to deal with the entire $\mathfrak{a}_{\mathbb{C}}^*$ at one time. But we can move step by step in the region containing the singularities and provide uniform estimates on the sets

$$\mathfrak{a}_{\mathbb{C}}^*(R) := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \operatorname{Re}\langle \lambda, \alpha\rangle < R \quad \text{for all } \alpha \in \Sigma^+\}, \tag{29}$$

for each $R > 0$. The λ-singularities of $\Phi(m; \lambda, a)$ are simple poles located on a locally finite family of complex hyperplanes and there are only finitely many singular hyperplanes intersecting $\mathfrak{a}_{\mathbb{C}}^*(R)$. The λ-singularities on $\mathfrak{a}_{\mathbb{C}}^*(R)$ are therefore canceled by multiplication by a polynomial function p_R which is a finite product of linear factors in $\langle \lambda, \alpha\rangle$ with $\alpha \in \Sigma^+$ (see Lemmas 4.4 and 4.5 below). To carry out the two distinguishing points, we shall also employ ideas of Wallach [42] and of van den Ban and Schlichtkrull [3]. As in [8], our final estimates hold for arbitrary (not necessarily geometric) multiplicities $m \in \mathcal{M}^+$. See also Remark 4.18 (c) below for comparisons with the estimates in [42] and [3].

We now enter into the details of the estimates. Recall that the Harish-Chandra series is a solution of the differential equation (9). The first important observation concerns the transformation of the operator $D(p_L, m) = L(m) + \langle \rho(m), \rho(m)\rangle$ under conjugation by $\delta(m/2)$.

Lemma 4.1. (see [14], Theorem 2.1.1) *Let $m \in \mathcal{M}$. Then the following equality of differential operators on A^+ holds:*

$$\delta(m/2) \circ D(p_L, m) \circ \delta(-m/2) = S(m), \tag{30}$$

where

$$S(m) := L_A + \sum_{\alpha\in\Sigma^+} \frac{m_\alpha(2 - m_\alpha - 2m_{2\alpha})}{(e^\alpha - e^{-\alpha})^2} \langle \alpha, \alpha\rangle$$

is the Schrödinger operator of the generalized Calogero-Moser system associated with $(\mathfrak{a}, \Sigma, m)$. $\square$

By (27) and by Lemma 4.1, the function $\Psi(m; \lambda, a)$ satisfies therefore the differential equation

$$S(m)\Psi(m; \lambda, a) = \langle \lambda, \lambda\rangle \Psi(m; \lambda, a), \qquad a \in A^+. \tag{31}$$

For any $T > 0$, we define $\mathfrak{a}^+(T) := \{H \in \mathfrak{a}^+ : \alpha_j(H) > T \quad \text{for all } j = 1, \ldots, l\}$ and set $A^+(T) := \exp(\mathfrak{a}^+(T))$. As before, $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ denotes the fundamental system of simple roots in Σ^+. On A^+ we have the series expansion

$$\delta\,(m/2; a) = e^{\rho(m)(\log a)} \prod_{\alpha\in\Sigma^+} \left(1 - e^{-2\alpha(\log a)}\right)^{m_\alpha/2} = e^{\rho(m)(\log a)} \sum_{\mu\in 2\Lambda} b_\mu(m) e^{-\mu(\log a)} \tag{32}$$

with $b_0(m) = 1$. The Cauchy product of (32) and (11) yields the series expansion (28) with

$$a_\mu(m; \lambda) := \sum_{\substack{\nu,\eta\in 2\Lambda\\ \nu+\eta=\mu}} b_\nu(m)\Gamma_\eta(m; \lambda)$$

and $a_0(m; \lambda) = 1$. The series converges absolutely in A^+ and uniformly in $A^+(T)$. Thus it can be differentiated term by term arbitrarily often. Let $\{H_j\}_{j=1}^l$ be an orthonormal basis in $\mathfrak{a}$ and set $\partial_j := \partial(H_j)$. Since

$$L_A e^{(\lambda-\mu)(\log a)} = \sum_{j=1}^{l} \partial_j^2 e^{(\lambda-\mu)(\log a)} = \sum_{j=1}^{l} (\lambda - \mu)(H_j)^2 e^{(\lambda-\mu)(\log a)} = \langle \lambda - \mu, \lambda - \mu\rangle e^{(\lambda-\mu)(\log a)},$$

we have for $a \in A^+(T)$

$$L_A \Psi(m; \lambda, a) = \sum_{\mu\in 2\Lambda} \langle \lambda - \mu, \lambda - \mu\rangle a_\mu(m; \lambda) e^{(\lambda-\mu)(\log a)}. \tag{33}$$

Noticing that

$$(e^\alpha - e^{-\alpha})^{-2} = \sum_{k=1}^{\infty} k e^{-2k\alpha}, \tag{34}$$

we obtain from substitution of (33) in the differential equation (31):

$$\sum_{\mu\in 2\Lambda} a_\mu(m;\lambda)\langle\lambda-\mu,\lambda-\mu\rangle e^{(\lambda-\mu)(\log a)} =$$

$$= \left[\sum_{\alpha\in\Sigma^+} \frac{m_\alpha(2-m_\alpha-2m_{2\alpha})}{(e^{\alpha(\log a)}-e^{-\alpha(\log a)})^2}\langle\alpha,\alpha\rangle + \langle\lambda,\lambda\rangle\right] \sum_{\mu\in 2\Lambda} a_\mu(m;\lambda)e^{(\lambda-\mu)(\log a)}$$

Since all series converge absolutely, we can multiply them out and collect the coefficients of the exponentials. This yields

$$\sum_{\mu\in 2\Lambda} a_\mu(m;\lambda)\langle\mu-2\lambda,\mu\rangle e^{(\lambda-\mu)(\log a)} =$$

$$= \sum_{\mu\in 2\Lambda}\sum_{\alpha\in\Sigma^+} m_\alpha(2-m_\alpha-2m_{2\alpha})\langle\alpha,\alpha\rangle \sum_{\substack{k\in\mathbb{N}\\ \mu-2k\alpha\in 2\Lambda}} k a_{\mu-2k\alpha}(m;\lambda)e^{(\lambda-\mu)(\log a)}.$$

Equating the coefficients of $e^{(\lambda-\mu)(\log a)}$ on both sides, we obtain the recursion relations

$$a_\mu(m;\lambda)\langle\mu-2\lambda,\mu\rangle = \sum_{\alpha\in\Sigma^+} m_\alpha(2-m_\alpha-2m_{2\alpha})\langle\alpha,\alpha\rangle \sum_{\substack{k\in\mathbb{N}\\ \mu-2k\alpha\in 2\Lambda}} k a_{\mu-2k\alpha}(m;\lambda) \tag{35}$$

for $\mu\in 2\Lambda\setminus\{0\}$, with initial condition $a_0(m;\lambda)=1$. They are a simplified version of the original relations (3.21) in [8].

Our goal is to use the recursion relations to determine growth estimates for the coefficients $a_\mu(m;\lambda)$. We first need a lemma, which is a local version of a lemma in [21]. We employ the usual multi-index notation: If $q=(q_1,\dots,q_l)\in\mathbb{N}_0^l$ denotes a multi-index and $z=(z_1,\dots,z_l)$ is an arbitrary point in $\mathbb{C}^l$, we write

$$z^q := z_1^{q_1}\cdots z_l^{q_l} \qquad \text{and} \qquad \partial^q := \frac{\partial^{q_1}}{\partial z^{q_1}}\cdots\frac{\partial^{q_l}}{\partial z^{q_l}}.$$

We moreover denote by $|z| := \left(\sum_{j=1}^l |z_j|^2\right)^{1/2}$ the norm of z associated with the natural Hermitian structure $(y,z) := \sum_{j=1}^l y_j\overline{z_j}$ in $\mathbb{C}^l$.

Lemma 4.2. (cf. [21], pp. 65–66)

(a) *Let p be a polynomial in the complex variable z of degree $\leq n$. Let $\varepsilon>0$ be fixed. Suppose f is a holomorphic function on a domain $D\subset\mathbb{C}$ with $D\supset\{z\in\mathbb{C}:|z|\leq\varepsilon\}$. Finally, let $\psi:[0,\infty)\to\mathbb{R}$ be a non-negative integrable function with compact support* $\operatorname{supp}\psi\subset[0,\varepsilon]$. *Then for all $k=0,1,\dots,n$ we have*

$$\left|f(0)\,\frac{d^kp}{dz^k}(0)\right| \int_{\mathbb{C}} |z|^k\,\psi(|z|)\,dz \leq C_{n,k}\int_{\mathbb{C}} |f(z)p(z)|\,\psi(|z|)\,dz,$$

where $C_{n,k} := \frac{n!}{(n-k)!}$. In the above integrals, dz denotes the Lebesgue measure on $\mathbb{C}$.

(b) *Let p be a polynomial in $z=(z_1,\dots,z_l)\in\mathbb{C}^l$ of degree n_j in the variable z_j for all $j=1,\dots,l$. For a fixed $\varepsilon>0$, let $r(\varepsilon)>0$ be chosen so that*

$$P_{r(\varepsilon)} := \{z\in\mathbb{C}^l : |z_j|\leq r(\varepsilon) \text{ for all } j=1,\dots,l\} \subset \{z\in\mathbb{C}^l : |z|<\varepsilon\}.$$

Suppose F is a holomorphic function on a domain $D\subset\mathbb{C}^l$ with $D\supset\{z\in\mathbb{C}^l:|z|<\varepsilon\}$. Finally, let $\Psi:\mathbb{C}^l\to\mathbb{R}$ be a non-negative integrable function depending only on $|z_1|,\dots,|z_l|$ and with compact support $\operatorname{supp}\Psi\subset P_{r(\varepsilon)}$. *Then for every multi-index $q=(q_1,\dots,q_l)$ with $q_j\leq n_j$ for all j, we have*

$$|F(0)\,\partial^q p(0)| \int_{\mathbb{C}^l} |z^q|\,\Psi(|z|)\,dz \leq C_{n,q}\int_{\mathbb{C}^l} |F(z)p(z)|\,\Psi(|z|)\,dz.$$

Here $C_{n,q} := \prod_{j=1}^{l} C_{n_j,q_j}$ *and* $dz = dz_1 \dots dz_n$ *is the Lebesgue measure on* $\mathbb{C}^l$.

(c) *Let* p, n, q, F, ε *and* $r(\varepsilon)$ *be as in part (b). Then there is a constant* $C_{\varepsilon,n,q} > 0$ *such that*

$$|F(0)\,\partial^q p(0)| \le C_{\varepsilon,n,q} \int_{P_{r(\varepsilon)}} |F(z)p(z)|\ dz. \tag{36}$$

□

As a corollary we obtain a variation of a classical lemma by Malgrange. It will play a decisive role at various stages.

Corollary 4.3. *Let* $U \subset \mathbb{C}^l$ *be any subset and let* $\varepsilon > 0$. *Consider the open* ε*-neighborhood*

$$U_\varepsilon := \{\zeta \in \mathbb{C}^l : \exists z \in U \quad \text{with} \quad |\zeta - z| < \varepsilon\}.$$

Let H *be a holomorphic function on* U_ε, *and let* p *be a polynomial such that* $F := H/p$ *is also holomorphic in* U_ε. *Let* $w_1, \dots, w_k$ *be elements in* $\mathbb{C}^l$ *having all equal norm* $|w_j| =: |w|$ $(j = 1, \dots, k)$. *Suppose there exists a constant* $C > 0$ *and* $N \in \mathbb{N}_0$ *such that*

$$|H(z)| \le C(1 + |z|)^N e^{\max_{j=1,\dots,k}(\operatorname{Re} z, w_j)}$$

for all $z \in U_\varepsilon$. *Then there is a constant* $C_\varepsilon > 0$ *such that*

$$|F(z)| \le C_\varepsilon (1 + |z|)^N e^{\max_{j=1,\dots,k}(\operatorname{Re} z, w_j) + \varepsilon |w|}$$

for all $z \in U$.

Proof. Let $z \in U$ be fixed and apply (36) to the holomorphic function $\zeta \mapsto F(\zeta + z)$ and the polynomial $\zeta \mapsto p(\zeta + z)$, which are defined on an open domain containing $\{\zeta \in \mathbb{C}^l : |\zeta| < \varepsilon\}$. Then $\zeta + z \in U_\varepsilon$, on which the estimates for $H = Fp$ are given. Let $r(\varepsilon)$ be as in Lemma 4.2, and let n_j denote the degree of p in the variable z_j. Since $r(\varepsilon) < \varepsilon$, we have for every multi-index $q = (q_1, \dots, q_n)$ with $q_j \le n_j$ for all j

$$\begin{aligned}
|F(z)\,\partial^q p(z)| &\le C_{\varepsilon,n,q} \int_{P_{r(\varepsilon)}} |F(\zeta + z)p(\zeta + z)|\ d\zeta \\
&\le C'_{\varepsilon,n,q} \sup_{|\zeta|<\varepsilon} |H(z + \zeta)| \\
&\le C'_{\varepsilon,n,q} \sup_{|\zeta|<\varepsilon} (1 + |z + \zeta|)^N e^{\max_{j=1,\dots,k}(\operatorname{Re}(z+\zeta), w_j)} \\
&\le C'_{\varepsilon,n,q} \sup_{|\zeta|<\varepsilon} (1 + |z| + |\zeta|)^N e^{\max_{j=1,\dots,k}(\operatorname{Re} z, w_j) + |\zeta||w|} \\
&\le C''_{\varepsilon,n,q} (1 + |z|)^N e^{\max_{j=1,\dots,k}(\operatorname{Re} z, w_j) + \varepsilon|w|}.
\end{aligned}$$

Since p is a polynomial, we can choose the multi-index q such that $\partial^q p(z)$ is a non-zero constant. This proves the required inequality on U. □

The norm $|\lambda| := \langle \lambda, \lambda \rangle^{1/2}$ on $\mathfrak{a}^*$ can be extended to $\mathfrak{a}^*_{\mathbb{C}}$ by setting $|\lambda| = \big(|\operatorname{Re}\lambda|^2 + |\operatorname{Im}\lambda|^2\big)^{1/2}$ for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, where $\operatorname{Re}\lambda, \operatorname{Im}\lambda \in \mathfrak{a}^*$ are defined pointwise on $\mathfrak{a}$. Similarly, we define the norm $|H| := \langle H, H \rangle^{1/2}$ on $\mathfrak{a}$ and extend it to $\mathfrak{a}_{\mathbb{C}}$. The fundamental system $\Pi = \{\alpha_1, \dots, \alpha_l\}$ is a basis for $\mathfrak{a}^*$. Define for $\lambda = \sum_{j=1}^{l} \lambda_j \alpha_j \in \mathfrak{a}^*$

$$\|\lambda\|_1 := \sum_{j=1}^{l} |\lambda_j|\,. \tag{37}$$

Then $\|\mu\|_1 = \ell(\mu)$ is the level of μ for all $\mu \in \Lambda$. By equivalence of all norms in $\mathfrak{a}^*$, there is a constant $Q \ge 1$ such that for all $\lambda \in \mathfrak{a}^*$

$$Q^{-1}\|\lambda\|_1 \le |\lambda| \le Q\|\lambda\|_1.$$

In particular, for $\mu \in \Lambda$ we deduce that

$$Q^{-1}\ell(\mu) \le |\mu| \le Q\ell(\mu). \tag{38}$$

The recursion relations (35) show that for $\mu \in 2\Lambda \setminus \{0\}$ each coefficient $a_\mu(m;\lambda)$ is a rational function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with at most simple poles along the hyperplanes

$$\mathcal{H}_\eta := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \langle \eta - 2\lambda, \eta\rangle = 0\}$$

for some $\eta \in 2\Lambda \setminus \{0\}$ with $\eta \le \mu$.

In the following R will always denote a *finite* positive real number and $\mathfrak{a}_{\mathbb{C}}^*(R)$ is as in (29).

Lemma 4.4. *Let $R > 0$ and $\eta \in 2\Lambda$, and let Q as in (38). If $\mathfrak{a}_{\mathbb{C}}^*(R) \cap \mathcal{H}_\eta \neq \emptyset$, then $\ell(\eta) < 2RQ^2$. Hence $\mathfrak{a}_{\mathbb{C}}^*(R)$ intersects at most finitely many hyperplanes $\mathcal{H}_\eta$ with $\eta \in 2\Lambda \setminus \{0\}$, and none if $R \le 1/Q^2$.*

Proof. Let $\nu \in \mathfrak{a}_{\mathbb{C}}^*(R) \cap \mathcal{H}_\eta$, so $\langle \eta,\eta\rangle - 2\langle \nu,\eta\rangle = 0$. Write $\eta = \sum_{j=1}^l \eta_j \alpha_j$. Then

$$\langle \eta,\eta\rangle = \mathrm{Re}\langle \eta,\eta\rangle = 2\,\mathrm{Re}\langle \nu,\eta\rangle = 2\sum_j \eta_j\,\mathrm{Re}\langle \nu,\alpha_j\rangle < 2R\ell(\eta).$$

Hence $\ell(\eta)^2 \le Q^2\langle \eta,\eta\rangle < 2RQ^2\ell(\eta)$, i.e. $\ell(\eta) < 2RQ^2$. For the last statement, notice that there are only finitely many $\eta \in 2\Lambda$ with $\ell(\eta) < 2RQ^2$ and that $\ell(\eta) \ge 2$ for $\eta \in 2\Lambda \setminus \{0\}$. Thus $2 \le \ell(\eta) < 2RQ^2$ implies $R > 1/Q^2$. □

For $R > 0$, we set

$$\mathcal{X}_R := \{\eta \in 2\Lambda \setminus \{0\} : \mathcal{H}_\eta \cap \mathfrak{a}_{\mathbb{C}}^*(2R) \neq 0\}. \tag{39}$$

Then $\mathcal{X}_R$ is a finite set by Lemma 4.4, and it is empty for $R \le 1/(2Q^2)$. We define the polynomial function p_R on $\mathfrak{a}_{\mathbb{C}}^*$ by

$$p_R(\lambda) := \prod_{\eta \in \mathcal{X}_R} \langle \eta - 2\lambda, \eta\rangle, \tag{40}$$

with the convention that $p_R(\lambda) \equiv 1$ when $\mathcal{X}_R = \emptyset$. We denote by $\deg p_R$ the degree of the polynomial p_R.

Lemma 4.5. *Suppose $R > 0$ and $\mu \in 2\Lambda$. Then:*

(a) *$p_R(\lambda)a_\mu(m;\lambda)$ is holomorphic in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(2R)$.*

(b) *For all $\nu < \mu$, the function $\dfrac{p_R(\lambda)a_\nu(m;\lambda)}{\langle \mu - 2\lambda, \mu\rangle}$ is holomorphic in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(2R)$.*

Proof. If $\mathcal{H}_\eta$ is a singular hyperplane for $a_\mu(m;\lambda)$ in $\mathfrak{a}_{\mathbb{C}}^*(2R)$, then $\mathcal{H}_\eta \cap \mathfrak{a}_{\mathbb{C}}^*(2R) \neq \emptyset$, hence $\eta \in \mathcal{X}_R$. The singularity along $\mathcal{H}_\eta$ is a simple pole, so it is removed by multiplication by $p_R(\lambda)$. This proves (a). For (b), observe first that, by the recursion relations, $\mathcal{H}_\mu$ is not a singular hyperplane for $a_\nu(m;\lambda)$ when $\nu < \mu$. The function $\frac{1}{\langle \mu-2\lambda,\mu\rangle}$ is singular in $\mathfrak{a}_{\mathbb{C}}^*(2R)$ if and only if $\mathcal{H}_\mu \cap \mathfrak{a}_{\mathbb{C}}^*(2R) \neq \emptyset$. In this case $\langle \mu - 2\lambda, \mu\rangle$ divides $p_R(\lambda)$, so $\frac{p_R(\lambda)a_\nu(m;\lambda)}{\langle \mu-2\lambda,\mu\rangle}$ is always holomorphic in $\mathfrak{a}_{\mathbb{C}}^*(2R)$. □

The following lemma is the classical argument of "ε-neighborhoods" from the analysis.

Lemma 4.6. *Suppose $R > 0$. Then there is $\varepsilon > 0$ with the property that*

$$\mathfrak{a}_{\mathbb{C}}^*(R)_\varepsilon := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \exists \nu \in \mathfrak{a}_{\mathbb{C}}^*(R) \quad \textit{with} \quad |\lambda - \nu| < \varepsilon\} \subset \mathfrak{a}_{\mathbb{C}}^*(2R).$$

□

Lemma 4.7. *Let $R' > R > 0$. Then there is a constant $C_{R,R',m} > 0$ with the following property: For every $\mu \in 2\Lambda$ with $\ell(\mu) \le R'$*

$$|p_R(\lambda)a_\mu(m;\lambda)| \le C_{R,R',m}(1 + |\lambda|)^{\deg p_R}$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$.

Proof. Let $\varepsilon > 0$ be as in Lemma 4.6. For $\mu \in 2\Lambda \setminus \{0\}$ with $\ell(\mu) \leq R'$, we set

$$\varepsilon(\mu) := \varepsilon \cdot \frac{R' + 1 - \ell(\mu)}{R' + 1}.$$

Then $0 < \varepsilon(\mu) \leq \varepsilon$, hence

$$\mathfrak{a}_{\mathbb{C}}^*(R)_{\varepsilon(\mu)} \subset \mathfrak{a}_{\mathbb{C}}^*(R)_{\varepsilon} \subset \mathfrak{a}_{\mathbb{C}}^*(2R). \tag{41}$$

By induction on the level $\ell(\mu) \leq R'$, we prove the existence of a constant $C_\mu \geq 0$ such that

$$|p_R(\lambda) a_\mu(m;\lambda)| \leq C_\mu (1 + |\lambda|)^{\deg p_R}$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)_{\varepsilon(\mu)}$. This is clear for $\mu = 0$. Assume it is true for all elements in 2Λ of level less than n ($< R'$) and prove it for $\mu \in 2\Lambda$ with $\ell(\mu) = n$. By the recursion relations (35), we have

$$|p_R(\lambda) a_\mu(m;\lambda)| \leq \sum_{\alpha \in \Sigma^+} m_\alpha \, |m_\alpha + 2m_{2\alpha} - 2| \, \langle \alpha, \alpha \rangle \sum_{\substack{k \in \mathbb{N} \\ \mu - 2k\alpha \in 2\Lambda}} k \left| \frac{p_R(\lambda) a_{\mu - 2k\alpha}(m;\lambda)}{\langle \mu - 2\lambda, \mu \rangle} \right|.$$

Suppose $\mu - 2k\alpha \in 2\Lambda$ for some $\alpha \in \Sigma^+$ and $k \in \mathbb{N}$. By inductive hypothesis, there is a constant $C_{\mu - 2k\alpha} \geq 0$ so that

$$|p_R(\lambda) a_{\mu-2k\alpha}(m;\lambda)| \leq C_{\mu-2k\alpha} (1 + |\lambda|)^{\deg p_R}$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)_{\varepsilon(\mu - 2k\alpha)}$. Here

$$\varepsilon(\mu - 2k\alpha) := \varepsilon \cdot \left(\frac{R' + 1 - \ell(\mu) + 2k\ell(\alpha)}{R' + 1} \right) \geq \varepsilon \cdot \left(\frac{R' - \ell(\mu) + 2}{R' + 1} \right) = \varepsilon(\mu) + \frac{\varepsilon}{R' + 1},$$

which implies together with (41)

$$\left[\mathfrak{a}_{\mathbb{C}}^*(R)_{\varepsilon(\mu)} \right]_{\varepsilon/(R'+1)} \subset \mathfrak{a}_{\mathbb{C}}^*(R)_{\varepsilon(\mu) + \varepsilon/(R'+1)} \subset \mathfrak{a}_{\mathbb{C}}^*(R)_{\varepsilon(\mu - 2k\alpha)} \subset \mathfrak{a}_{\mathbb{C}}^*(2R).$$

By part (b) of Lemma 4.5, the function $\dfrac{p_R(\lambda) a_{\mu-2k\alpha}(m;\lambda)}{\langle \mu - 2\lambda, \mu \rangle}$ is holomorphic in $\mathfrak{a}_{\mathbb{C}}^*(2R)$. Corollary 4.3 then implies the existence of a constant $C'_{\mu - 2k\alpha} \geq 0$ such that

$$\left| \frac{p_R(\lambda) a_{\mu-2k\alpha}(m;\lambda)}{\langle \mu - 2\lambda, \mu \rangle} \right| \leq C'_{\mu-2k\alpha} (1 + |\lambda|)^{\deg p_R}$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)_{\varepsilon(\mu)}$. The inductive step is therefore concluded when we set

$$C_\mu := \sum_{\alpha \in \Sigma^+} m_\alpha \, |m_\alpha + 2m_{2\alpha} - 2| \, \langle \alpha, \alpha \rangle \sum_{\substack{k \in \mathbb{N} \\ \mu - 2k\alpha \in 2\Lambda}} k C'_{\mu - 2k\alpha}.$$

This also proves the lemma, since there are only finitely many $\mu \in 2\Lambda$ satisfying $\ell(\mu) \leq R'$. ∎

As in [42], we now define for $c \in [0, \infty)$

$$\delta_c(m) := e^{c\rho(m)} \delta(-cm/2) = \prod_{\alpha \in \Sigma^+} \left(1 - e^{-2\alpha} \right)^{-cm_\alpha/2}. \tag{42}$$

Lemma 4.8. *For every fixed $c \in \mathbb{R}$, we have the series expansion on A^+*

$$\delta_c(m) = \sum_{\mu \in 2\Lambda} d_\mu(m;c) e^{-\mu} \tag{43}$$

with $d_0(m;c) = 1$. The series converges absolutely in A^+ and uniformly in $A^+(T)$ for all $T > 0$.

If we assume $c \in (0, \infty)$, then $d_\mu(m;c) > 0$ for all $\mu \in 2\Lambda$.

Proof. For $r \in \mathbb{R}$ and $n \in \mathbb{N}_0$, we adopt the familiar notation $\binom{r}{n} := \prod_{h=1}^{n} \frac{r-h+1}{h}$ for the binomial coefficients. As usual, empty products are defined to be 1. Recall that the binomial series $(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k$ converges absolutely in $|x| < 1$ and uniformly in $|x| \leq S < 1$. Therefore for $\alpha \in \Sigma^+$ and $a \in A^+$, we have

$$\left(1 - e^{-2\alpha(\log a)}\right)^{-cm_\alpha/2} = \sum_{k=0}^{\infty} \binom{-cm_\alpha/2}{k} (-1)^k e^{-2k\alpha(\log a)}, \tag{44}$$

with absolute convergence in A^+ and uniform convergence in $A^+(T)$ for all $T > 0$. Write $\Sigma^+ = \{\beta_1, \ldots, \beta_n\}$. Cauchy products yield

$$\prod_{\alpha \in \Sigma^+} \left(1 - e^{-2\alpha(\log a)}\right)^{-cm_\alpha/2} = \sum_{\mu \in 2\Lambda} d_\mu(m;c) e^{-\mu(\log a)},$$

where

$$d_\mu(m;c) := \sum_{\substack{k_1,\ldots,k_n \in \mathbb{N}_0 \\ 2(k_1\beta_1 + \cdots + k_n\beta_n) = \mu}} (-1)^{k_1 + \cdots + k_n} \binom{-cm_{\beta_1}/2}{k_1} \cdots \binom{-cm_{\beta_n}/2}{k_n}.$$

As well as each of the series (44), also their Cauchy product converges absolutely in A^+ and uniformly in $A^+(T)$ for all $T > 0$. Observe that $d_0(m;c) = 1$.

If $x > 0$, then

$$(-1)^k \binom{-x}{k} := (-1)^k \prod_{h=1}^{k} \left(\frac{-x-h+1}{h}\right) = \prod_{h=1}^{k} \frac{x+h-1}{h} > 0.$$

Suppose then $c \in (0,\infty)$. Since we are assuming $m_\alpha > 0$ for all $\alpha \in \Sigma^+$, we have $(-1)^k \binom{-cm_\alpha/2}{k} > 0$ for all $\alpha \in \Sigma^+$ and $k \in \mathbb{N}_0$. In this case we therefore obtain $d_\mu(m;c) > 0$ for all $\mu \in 2\Lambda$. □

The idea for estimating the modified Harish-Chandra series $\Psi(m;\lambda,a)$ is the following. Suppose $c \in [0,\infty)$. We determine a differential equation satisfied by $\delta_c(m)$ and, by formal insertion of the series (43), we obtain recursion relations for the coefficients $d_\mu(m;c)$. These recursion relations can be used to compare $|a_\mu(m,\lambda)|$ and $d_\mu(m;c)$ for the same $\mu \in 2\Lambda$. This provides an estimate for $\left|e^{-\lambda}\Psi(m;\lambda)\right|$ via $\delta_c(m)$. A suitable value of the parameter c is determined along the way.

Lemma 4.9. *For all $m \in \mathcal{M}^+$, $c \in [0,\infty)$ and $H \in \mathfrak{a}$, the function $\delta_c(m)$ satisfies the differential equation on A^+*

$$\begin{aligned}\big(L_A - \partial(H)\big)\delta_c(m) &= \\ &= c \sum_{\alpha \in \Sigma^+} \left[m_\alpha(\coth\alpha - 1)\left(\frac{\alpha(H)}{2} - c\langle\rho(m),\alpha\rangle\right) + \frac{m_\alpha(cm_\alpha + 2cm_{2\alpha} + 2)}{(e^\alpha - e^{-\alpha})^2}\langle\alpha,\alpha\rangle\right]\delta_c(m).\end{aligned}$$

Proof. Let $m \in \mathcal{M}^+$ and $c \in [0,\infty)$ be fixed. For brevity, we omit the dependence on m in the following computations. Observe that for all $H \in \mathfrak{a}$

$$\delta^{-1}\partial(H)\delta = \sum_{\alpha \in \Sigma^+} m_\alpha \alpha(H) \coth\alpha,$$

$$\partial(H)\left(\delta^{-1}\partial(H)\delta\right) = -4 \sum_{\alpha \in \Sigma^+} m_\alpha \alpha(H)^2 \frac{1}{(e^\alpha - e^{-\alpha})^2}.$$

Hence from

$$\delta_c^{-1}\partial(H)\delta_c = c\rho(H) - \frac{c}{2}\,\delta^{-1}\partial(H)\delta,$$

$$\delta_c^{-1}\partial(H)^2\delta_c = \left[c\rho(H) - \frac{c}{2}\,\delta^{-1}\partial(H)\delta\right]^2 - \frac{c}{2}\,\partial(H)\left(\delta^{-1}\partial(H)\delta\right),$$

we obtain for $L_A = \sum_{j=1}^{l} \partial_j^2$

$$\delta_c^{-1} L_A(\delta_c) = c^2\langle\rho,\rho\rangle + \frac{c^2}{4}\sum_{\alpha,\beta\in\Sigma^+} m_\alpha m_\beta \langle\alpha,\beta\rangle \coth\alpha \coth\beta$$
$$- c^2 \sum_{\alpha\in\Sigma^+} m_\alpha\langle\rho,\alpha\rangle\coth\alpha + 2c\sum_{\alpha\in\Sigma^+} m_\alpha\langle\alpha,\alpha\rangle\frac{1}{(e^\alpha - e^{-\alpha})^2}.$$

The equality

$$\frac{1}{4}\sum_{\alpha,\beta\in\Sigma^+} m_\alpha m_\beta\langle\alpha,\beta\rangle\coth\alpha\coth\beta = \langle\rho,\rho\rangle + \sum_{\alpha\in\Sigma^+}\frac{m_\alpha(m_\alpha + 2m_{2\alpha})}{(e^\alpha - e^{-\alpha})^2}\langle\alpha,\alpha\rangle$$

is proven on p. 21 in [14]. Therefore, writing

$$\langle\rho,\rho\rangle = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha\langle\rho,\alpha\rangle,$$

we conclude that

$$\delta_c^{-1} L_A(\delta_c) = c^2\sum_{\alpha\in\Sigma^+} m_\alpha\langle\rho,\alpha\rangle(1-\coth\alpha) + c\sum_{\alpha\in\Sigma^+}\frac{m_\alpha(cm_\alpha + 2cm_{2\alpha} + 2)}{(e^\alpha - e^{-\alpha})^2}\langle\alpha,\alpha\rangle.$$

The claim then follows because for all $H \in \mathfrak{a}$ we have

$$-\delta_c^{-1}\partial(H)\delta_c = -c\rho(H) + \frac{c}{2}\delta^{-1}\partial(H)\delta = \frac{c}{2}\sum_{\alpha\in\Sigma^+} m_\alpha(\coth\alpha - 1)\alpha(H).$$

□

Lemma 4.10. *The coefficients $d_\mu(m;c)$ of the series (43) satisfy the recurrence relations*

$$\big(\langle\mu,\mu\rangle + \mu(H)\big)d_\mu(m;c) = \sum_{\alpha\in\Sigma^+}\sum_{\substack{k\in\mathbb{N}\\ \mu-2k\alpha\in 2\Lambda}}\bigg[2cm_\alpha\left(\frac{\alpha(H)}{2} - c\langle\rho(m),\alpha\rangle\right) +$$
$$+ kcm_\alpha(cm_\alpha + 2cm_{2\alpha} + 2)\langle\alpha,\alpha\rangle\bigg] d_{\mu-2k\alpha}(m;c)$$

for $\mu \in 2\Lambda\setminus\{0\}$, and $d_0(m;c) = 1$.

Proof. We formally insert the series (43) in the differential equation of Lemma 4.9. By termwise differentiation of the series (43) we obtain

$$\big(L_A - \partial(H)\big)(\delta_c) = \sum_{\mu\in\Lambda} d_\mu(m;c)\big(\langle\mu,\mu\rangle + \mu(H)\big)e^{-\mu}.$$

From (34) follows

$$\frac{1}{(e^\alpha - e^{-\alpha})^2}\delta_c = \sum_{\mu\in 2\Lambda}\sum_{\substack{k\in\mathbb{N}\\ \mu-2k\alpha\in 2\Lambda}} kd_{\mu-2k\alpha}(m;c)e^{-\mu}.$$

Moreover,

$$(\coth\alpha - 1)\delta_c = 2\sum_{\mu\in 2\Lambda}\sum_{\substack{k\in\mathbb{N}\\ \mu-2k\alpha\in 2\Lambda}} d_{\mu-2k\alpha}(m;c)e^{-\mu}.$$

The recursion relations are then obtained by equating the coefficients of $e^{-\mu}$ in the resulting equation. □

Lemma 4.11. *There exist constants $0 \le c < 1$ and $r > 1$ such that the inequality*

$$c(cm_\alpha + 2cm_{2\alpha} + 2) \ge r\,|m_\alpha + 2m_{2\alpha} - 2|$$

holds for all $\alpha \in \Sigma^+$.

Proof. (cf. [42], p. 782) Let $r > 1$ be arbitrarily fixed. For all $\alpha \in \Sigma^+$, define

$$c_{\alpha,r} := \frac{-1 + \sqrt{1 + r(m_\alpha + 2m_{2\alpha})\,|m_\alpha + 2m_{2\alpha} - 2|}}{m_\alpha + 2m_{2\alpha}}.$$

Notice that $c_{\alpha,r} \geq 0$. Set $c_r := \max_{\alpha \in \Sigma^+} c_{\alpha,r}$. Then $c_r \geq 0$. Moreover,

$$c_r^2(m_\alpha + 2m_{2\alpha}) + 2c_r - r\,|m_\alpha + 2m_{2\alpha} - 2| \geq 0 \tag{45}$$

for all $\alpha \in \Sigma^+$. Indeed $c_{\alpha,r}$ is the largest root of the quadratic equation $(m_\alpha + 2m_{2\alpha})x^2 + 2x - r\,|m_\alpha + 2m_{2\alpha} - 2| = 0$.

We now want to show that we can select $r > 1$ for which also $c_r < 1$ holds. Suppose $\beta \in \Sigma^+$ (depending on r) satisfies $c_r = c_{\beta,r}$. Then the inequality $c_r < 1$ can be written as

$$\frac{-1 + \sqrt{1 + r(m_\beta + m_{2\beta})\,|m_\beta + 2m_{2\beta} - 2|}}{m_\beta + 2m_{2\beta}} < 1.$$

It is satisfied precisely when r fulfills

$$1 + r(m_\beta + 2m_{2\beta})\,|m_\beta + 2m_{2\beta} - 2| < (m_\beta + 2m_{2\beta} + 1)^2. \tag{46}$$

Observe that $(m_\beta + 2m_{2\beta} + 1)^2 - 1 = (m_\beta + 2m_{2\beta})(m_\beta + 2m_{2\beta} + 2)$. Therefore condition (46) is equivalent to

$$r\,|m_\beta + 2m_{2\beta} - 2| < m_\beta + 2m_{2\beta} + 2. \tag{47}$$

Since $|m_\alpha + 2m_{2\alpha} - 2| < m_\alpha + 2m_{2\alpha} + 2$ for all $\alpha \in \Sigma^+$, there exist r's so that

$$1 < r < \min_{\alpha \in \Sigma^+} \frac{m_\alpha + 2m_{2\alpha} + 2}{|m_\alpha + 2m_{2\alpha} - 2|},$$

and any of them satisfies (47). Let $\mathcal{R}$ denote the set of such r's. Then any c_r with $r \in \mathcal{R}$ satisfies all the requirements. □

Lemma 4.12. *Let $m \in \mathcal{M}^+$ be a multiplicity function. Then the system of inequalities of Lemma 4.11 admits the solution $c = 0$ if and only if $m_\alpha + 2m_{2\alpha} - 2 = 0$ for all $\alpha \in \Sigma^+$. In this case, $a_\mu(m;\lambda) = 0$ for all $\mu \in 2\Lambda \setminus \{0\}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.*

Proof. The first statement is obvious. Let $\mu \in 2\Lambda \setminus \{0\}$ be fixed. The recursion relations (35) define $a_\mu(m;\lambda)$ as a rational function of the variable $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Suppose $c = 0$. Then $m_\alpha + 2m_{2\alpha} - 2 = 0$ for all $\alpha \in \Sigma^+$, and the right-hand side of (35) vanishes. A rational function is identically zero when it vanishes on the complement of a lower dimensional manifold. The equality $a_\mu(m;\lambda)\langle \mu - 2\lambda, \mu\rangle = 0$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ therefore implies $a_\mu(m;\lambda) = 0$. □

Remark 4.13. Suppose m is geometric. Recall that we are assuming $m_\alpha > 0$ for $\alpha \in \Sigma$. If $m_{2\alpha} \neq 0$, then $2m_{2\alpha} - 2 \geq 0$. Hence the equality $m_\alpha + 2m_{2\alpha} - 2 = 0$ can be satisfied if and only if $2\alpha \notin \Sigma$ and $m_\alpha = 2$.

Fix $c \in [0,1)$ and $r > 1$ satisfying the inequalities of Lemma 4.11. We choose $H \in \mathfrak{a}$ such that

$$\alpha(H) \geq \max\{2c\langle \rho(m), \alpha\rangle, 0\} \quad \text{for all } \alpha \in \Sigma^+. \tag{48}$$

Condition (48) ensures that the first summands in Lemma 4.10 are all positive and that $\mu(H) \geq 0$ for all $\mu \in 2\Lambda$.

Let $R > 0$. We wish to determine, for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$ and $\mu \in 2\Lambda$, an upper estimate for $|a_\mu(m;\lambda)|$ by the coefficients $d_\mu(m;c)$ of the series expansion of δ_c. For finitely many $\mu \in 2\Lambda$ this can be easily done by means of Lemma 4.7.

Lemma 4.14. *Let $c \in [0,1)$ and $r > 1$ satisfy the inequalities of Lemma 4.11. Suppose $R' > R > 0$. Then there is a constant $C_{R,R',m,c,r} > 0$ with the following property: For every $\mu \in 2\Lambda$ with $\ell(\mu) \leq R'$*

$$|p_R(\lambda) a_\mu(m;\lambda)| \leq C_{R,R',m,c,r}\, d_\mu(m;c)(1 + |\lambda|)^{\deg p_R}$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$.

Proof. If $c = 0$, then $d_\mu(m; c) = 0$ for all $\mu \in 2\Lambda \setminus \{0\}$. But then also $a_\mu(m; \lambda) = 0$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $\mu \in 2\Lambda \setminus \{0\}$ by Lemma 4.12. The inequality is therefore trivially satisfied.

Suppose now that $c \in (0, 1)$. Lemma 4.8 ensures then that $d_\mu(m; c) > 0$ for all $\mu \in 2\Lambda \setminus \{0\}$. Set

$$d := \min\{d_\mu(m; c) : \mu \in 2\Lambda, \ell(\mu) \leq R'\}.$$

Then $d > 0$ and the required inequality is satisfied with $C_{R,R',m,c,r} := C_{R,R',m}/d$, where $C_{R,R',m}$ is the constant of Lemma 4.7. $\square$

To extend the inequalities of Lemma 4.14 to all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and *all* $\mu \in 2\Lambda$, we compare the recursion relations of $a_\mu(m; \lambda)$ and $d_\mu(m; c)$; see (35) and Lemma 4.10, respectively. Looking at these recursion relations, we realize that, when H is chosen as in (48), we need an inequality of the form

$$|\langle \mu, \mu \rangle - 2\langle \lambda, \mu \rangle| > \frac{1}{r} \left[\langle \mu, \mu \rangle + \mu(H)\right]$$

to hold for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$ and all $\mu \in 2\Lambda$. Such an inequality is clearly not possible because its right-hand side is always positive, whereas its left-hand side may be equal to zero. Nevertheless, the next lemma states that this inequality is satisfied by all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$ and by all $\mu \in 2\Lambda \setminus D_{R,r}$, where $D_{R,r}$ contains only finitely many elements. This will be enough, because for finitely many $\mu \in 2\Lambda$ we can apply Lemma 4.14.

Lemma 4.15. *Let $H \in \mathfrak{a}$ be chosen as in (48). For $r > 1$ and $R > 0$ we set*

$$D_{R,r} := \{\mu \in 2\Lambda : \langle \mu, \mu \rangle + \mu(H) \geq r \left[\langle \mu, \mu \rangle - 2R\ell(\mu)\right]\}.$$

Then $D_{R,r}$ is a finite set. More precisely,

$$D_{R,r} \subset \{\mu \in 2\Lambda : \ell(\mu) \leq R_r\}$$

with

$$R_r := \frac{Q}{r-1} \left(|H| + 2rRQ\right) \tag{49}$$

and Q as in (38). Furthermore,

$$|\langle \mu, \mu \rangle - 2\langle \lambda, \mu \rangle| > \frac{1}{r} \left[\langle \mu, \mu \rangle + \mu(H)\right].$$

for all $\mu \in 2\Lambda \setminus D_{R,r}$ and all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$.

Proof. With Q as in (38) and $\mu \in D_{R,r}$, we have

$$(r-1)\langle \mu, \mu \rangle \leq \mu(H) + 2rR\ell(\mu) \leq |\mu| \, |H| + 2rRQ \, |\mu|.$$

Hence $(r-1) \, |\mu| \leq |H| + 2rRQ$, which implies

$$\ell(\mu) \leq Q \, |\mu| \leq \frac{Q}{r-1} \left(|H| + 2rRQ\right) =: R_r.$$

Notice that R_r is finite because $r > 1$. Thus $D_{R,r} \subset \{\mu \in 2\Lambda : \ell(\mu) \leq R_r\}$ is a finite set.

Suppose $\mu \in 2\Lambda \setminus D_{R,r}$. Then

$$\langle \mu, \mu \rangle + \mu(H) < r \left[\langle \mu, \mu \rangle - 2R\ell(\mu)\right].$$

This implies in particular that $\langle \mu, \mu \rangle - 2R\ell(\mu) > 0$ because of (48) and because $\langle \mu, \mu \rangle > 0$ when $\ell(\mu) > 0$. Write $\mu = \sum_{j=1}^{l} \mu_j \alpha_j$ with $\mu_j \in 2\mathbb{N}_0$. Then for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$ we have

$$\begin{aligned}
|\langle \mu, \mu \rangle - 2\langle \lambda, \mu \rangle| &\geq \operatorname{Re}\left[\langle \mu, \mu \rangle - 2\langle \lambda, \mu \rangle\right] \\
&= \langle \mu, \mu \rangle - 2\sum_j \mu_j \operatorname{Re}\langle \lambda, \alpha_j \rangle \\
&> \langle \mu, \mu \rangle - 2R\big(\sum_j \mu_j\big) \\
&= \langle \mu, \mu \rangle - 2R\ell(\mu) \\
&> \frac{1}{r}\left[\langle \mu, \mu \rangle + \mu(H)\right].
\end{aligned}$$

□

Lemma 4.16. *Let $R > 0$ and let p_R be the polynomial defined in (40). Suppose $c \in [0, 1)$ is chosen as in Lemma 4.11 for some $r > 1$. Then there is a constant $C_{R,c,m} > 0$ such that*

$$|p_R(\lambda) a_\mu(m; \lambda)| \leq C_{R,c,m} d_\mu(m; c)(1 + |\lambda|)^{\deg p_R}$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$ and all $\mu \in 2\Lambda$.

Proof. We proceed by induction on $\ell(\mu)$. For a fixed R, let R_r be as in (49). Observe that $R_r > R$. Since the set of $\mu \in 2\Lambda$ with $\ell(\mu) \leq R_r$ is finite, the claim is true for all such μ by Lemma 4.14. Assume the claim as true for all $\eta \in 2\Lambda$ so that $\ell(\eta) < n$, and prove it for $\mu \in 2\Lambda$ with $\ell(\mu) = n$ (and we can assume $n > R_r$). Since $\ell(\mu) > R_r$, then $\mu \notin D_{R,r}$. Hence, by Lemma 4.15,

$$\frac{1}{|\langle \mu, \mu \rangle - 2\langle \lambda, \mu \rangle|} < \frac{r}{\langle \mu, \mu \rangle + \mu(H)}.$$

From the recursion relations (35), the inductive hypothesis, Lemma 4.11 and (48), we conclude

$$\begin{aligned}
&|p_R(\lambda) a_\mu(m; \lambda)| \leq \\
&\leq \frac{r}{\langle \mu, \mu \rangle + \mu(H)} \sum_{\alpha \in \Sigma^+} m_\alpha \, |m_\alpha + 2m_{2\alpha} - 2| \, \langle \alpha, \alpha \rangle \sum_{\substack{k \in \mathbb{N} \\ \mu - 2k\alpha \in 2\Lambda}} k \, |p_R(\lambda) a_{\mu - 2k\alpha}(m; \lambda)| \\
&\leq \frac{C_{R,c,m}}{\langle \mu, \mu \rangle + \mu(H)} \Big[\sum_{\alpha \in \Sigma^+} m_\alpha (r \, |2 - m_\alpha - 2m_{2\alpha}|) \langle \alpha, \alpha \rangle \sum_{\substack{k \in \mathbb{N} \\ \mu - 2k\alpha \in 2\Lambda}} k d_{\mu - 2k\alpha}(m; c) \Big] (1 + |\lambda|)^{\deg p_R} \\
&\leq \frac{C_{R,c,m}}{\langle \mu, \mu \rangle + \mu(H)} \Big[\sum_{\alpha \in \Sigma^+} c m_\alpha (c m_\alpha + 2m_{2\alpha} + 2) \langle \alpha, \alpha \rangle \sum_{\substack{k \in \mathbb{N} \\ \mu - 2k\alpha \in 2\Lambda}} k d_{\mu - 2k\alpha}(m; c) \Big] (1 + |\lambda|)^{\deg p_R} \\
&\leq \frac{C_{R,c,m}}{\langle \mu, \mu \rangle + \mu(H)} \Big\{ \sum_{\alpha \in \Sigma^+} \sum_{\substack{k \in \mathbb{N} \\ \mu - 2k\alpha \in 2\Lambda}} \Big[2c m_\alpha \Big(\frac{\alpha(H)}{2} - c\langle \rho(m), \alpha \rangle \Big) + \\
&\qquad\qquad + k c m_\alpha (c m_\alpha + 2c m_{2\alpha} + 2) \langle \alpha, \alpha \rangle \Big] d_{\mu - 2k\alpha}(m; c) \Big\} (1 + |\lambda|)^{\deg p_R} \\
&= C_{R,c,m} d_\mu(m; c)(1 + |\lambda|)^{\deg p_R}.
\end{aligned}$$

The last equality is provided by the recursion relations for $d_\mu(m; c)$ in Lemma 4.10. This completes the inductive step. □

Theorem 4.17. *Let $R > 0$ and let p_R be the polynomial defined in (40). Suppose $m \in \mathcal{M}^+$. Then there exist $c \in [0, 1)$ (depending only on the multiplicity function m) and $C_{R,c,m} > 0$ such that*

$$|p_R(\lambda) \Psi(m; \lambda, a)| \leq C_{R,c,m} \delta(m; a)^{-c/2} (1 + |\lambda|)^{\deg p_R} e^{(c\rho(m) + \operatorname{Re} \lambda)(\log a)} \tag{50}$$

and

$$\left|p_R(\lambda)\delta(m;a)^{(c+1)/2}\Phi(m;\lambda,a)\right| \leq C_{R,c,m}(1+|\lambda|)^{\deg p_R} e^{(c\rho(m)+\operatorname{Re}\lambda)(\log a)} \tag{51}$$

for all $a \in A^+$ *and* $\lambda \in \mathfrak{a}_\mathbb{C}^*(R)$.

We can choose $p_R \equiv 1$ *for all sufficiently small* $R > 0$ *(e.g.* $R \leq 1/(2Q^2)$ *with* Q *as in (38)).*

Proof. Because of Lemma 4.16

$$\begin{aligned}
|p_R(\lambda)\Psi(m;\lambda,a)| &\leq \sum_{\mu\in 2\Lambda} |p_R(\lambda)a_\mu(m;\lambda)| \left|e^{(\lambda-\mu)(\log a)}\right| \\
&\leq C_{R,c,m} \sum_{\mu\in 2\Lambda} d_\mu(m;c)e^{-\mu(\log a)}(1+|\lambda|)^{\deg p_R} e^{\operatorname{Re}\lambda(\log a)} \\
&\leq C_{R,c,m}\delta_c(m;a)(1+|\lambda|)^{\deg p_R} e^{\operatorname{Re}\lambda(\log a)}.
\end{aligned}$$

The first inequality therefore follows from (43), and the second from the definition (27) of Ψ. □

Remark 4.18. (a) Two features of the estimates (51) will be important in the following discussion:

i. The condition $c \in [0,1)$ implies $(c+1)/2 \in [1/2,1)$;
ii. Each $\mathfrak{a}_\mathbb{C}^*(R)$ contains a W_Θ-invariant open subset $\mathfrak{a}_{\mathbb{C},\Theta}^*(R)$ such that the set of the $\mathfrak{a}_{\mathbb{C},\Theta}^*(R)$ for $R > 0$ covers $\mathfrak{a}_\mathbb{C}^*$.

(b) For all $\lambda \in \mathfrak{a}_\mathbb{C}^*$ the factor $\delta(m;a)^{(c+1)/2}$ cancels the possible singularities of $\Phi(m;\lambda,a)$ along the walls of A^+. The exact asymptotic behavior of $\Phi(m;\lambda,a)$ along the walls is in general not known. In the complex and rank-one cases, these asymptotics can be deduced by the explicit formulas for the Harish-Chandra series. We shall compare these exact asymptotics with our estimate in Examples 4.19 and 4.20 below.

(c) For $0 < R < 1/(2Q^2)$ we have $p_R(\lambda) = 1$ for all λ. Hence the estimate (51) reduces to

$$\delta(m;a)^{(c+1)/2}\,|\Phi(m;\lambda,a)| \leq C_{R,c,m} e^{(c\rho(m)+\operatorname{Re}\lambda)(\log a)} \tag{52}$$

for all $a \in A^+$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*(R)$. Only for geometric multiplicities m and for $\lambda \in \mathfrak{a}_\mathbb{C}^*$ satisfying $\operatorname{Re}\langle\lambda,\alpha\rangle \leq 0$ for all $\alpha \in \Sigma^+$, the estimate (52) has been established by Wallach [42]. Our optimal (=smallest) value for c coincides with the one determined in [42] despite the use of different reductions for the recursion relations. The reduction methods used in Wallach's paper cannot be generalized to non-geometric multiplicities because they are based on Araki's tables of multiplicities.

Estimates of the type (51) have been determined in the more general case of Eisenstein integrals by van den Ban and Schlichtkrull in [3], where the exponent of δ is however left undetermined (it is also not stated whether it is smaller than 1).

(d) We introduce the notation $A^+ \ni a \to +\infty$ to indicate that $a \in A^+$ and that $\alpha(\log a) \to +\infty$ for all $\alpha \in \Sigma^+$, i.e. $\log a \in \mathfrak{a}^+$ moves towards infinity in such a way that also its distance from each of the walls of the Weyl chamber $\mathfrak{a}^+$ goes to infinity. Since for $A^+ \ni a \to +\infty$

$$\delta(m;a)^{(c+1)/2} = e^{(c+1)\rho(m)(\log a)} \prod_{\alpha\in\Sigma^+} (1-e^{-2\alpha(\log a)})^{(c+1)m_\alpha/2} \sim e^{(c+1)\rho(m)(\log a)}$$

$$\Phi(m;\lambda,a) \sim e^{(\lambda-\rho(m))(\log a)},$$

we conclude that, for a certain constant $C_{\lambda,R} > 0$,

$$\left|p_R(\lambda)\delta(m;a)^{(c+1)/2}\Phi(m;\lambda,a)\right| \sim C_{\lambda,R}\, e^{(c\rho(m)+\operatorname{Re}\lambda)(\log a)}.$$

This shows that the estimate (51) is sharp for $A^+ \ni a \to +\infty$.

(e) For $R > (1/2Q^2)$ the exponent $\deg p_R$ does not give a sharp estimate in $\lambda \in \mathfrak{a}_\mathbb{C}^*(R)$. The first a-priori reason comes from Theorem 1.4, which states that the λ-singularities of $\Phi(m;\lambda,a)$ are much fewer than those suggested by the recursion relations. Moreover, the estimates on $\mathfrak{a}_\mathbb{C}^*(R)$ have been obtained by means of p_R, which is the polynomial canceling all first-order singularities along the hyperplanes $\mathcal{H}_\eta$ intersecting the larger domain $\mathfrak{a}_{\mathbb{C}}^*(2R)$. We couldn't have replaced p_R by the polynomial $p_{R/2}$ (which is the one canceling the λ-singularities along the hyperplanes intersecting $\mathfrak{a}_{\mathbb{C}}^*(R)$) because of the application of Corollary 4.3 in Lemma 4.7. Indeed, Corollary 4.3 requires holomorphy on some ε-neighborhood of $\mathfrak{a}_{\mathbb{C}}^*(R)$. For many applications (e.g. inversion formulas and Paley-Wiener type theorems) the important feature of the estimate in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$ is its form $(1+|\lambda|)^{d(R)} e^{\operatorname{Re}\lambda}$, and not the exact value of the (non-negative) exponent $d(R)$. Since this form can be achieved in any case, our choice of working on $\mathfrak{a}_{\mathbb{C}}^*(2R)$ (rather than on $\mathfrak{a}_{\mathbb{C}}^*(R+\varepsilon)$ for some $\varepsilon > 0$) has only been dictated by the attempt of simplifying the notation.

Example 4.19 (The complex case)**.** If Σ is reduced and $m_\alpha = 2$ for all $\alpha \in \Sigma$, then the right-hand side of the inequality in Lemma 4.11 vanishes. Thus our optimal value of c is in this case $c = 0$ (obtained for any arbitrary $r > 1$). The explicit form of the Harish-Chandra series $\Phi(2;\lambda,a) = \Delta(a)^{-1} e^{\lambda(\log a)}$ (see Example 1.9) shows that we can choose $p_R \equiv 1$ for all $R > 0$. Since $\delta(2)^{1/2} = \delta(1) = \Delta$, the estimate (51) reduces with $c = 0$ to

$$\Delta(a)\, |\Phi(2;\lambda,a)| \le C_R (1+|\lambda|)^{\deg p_R} e^{\operatorname{Re}\lambda(\log a)}$$

for all $a \in A^+$. For $R > 0$ small enough, Theorem 4.17 yields $\deg p_R = 0$. In this case the estimate is the best possible. For arbitrary $R > 0$ the values of $\deg p_R$ found via Theorem 4.17 are positive, and the term $(1+|\lambda|)^{\deg p_R}$ is in fact superfluous.

Example 4.20 (The rank-one case)**.** In the rank-one case, the system of inequalities in Lemma 4.11 consists of only one inequality when $2\alpha \notin \Sigma^+$, and of two when $2\alpha \in \Sigma^+$. We now determine the infimum c_0 of all possible $c \in [0,1)$ which can be used in the estimate (51) for $\Phi(m;\lambda,t)$. Recall our convention that $m_{2\alpha} = 0$ is equivalent to $2\alpha \notin \Sigma$. Using the notation introduced in the proof of Lemma 4.11, we have

$$\inf_{r>1} c_{\alpha,r} = \frac{-1+\sqrt{1+(m_\alpha+2m_{2\alpha})\,|m_\alpha+2m_{2\alpha}-2|}}{m_\alpha+2m_{2\alpha}},$$
$$\inf_{r>1} c_{2\alpha,r} = \frac{-1+\sqrt{1+m_{2\alpha}\,|m_{2\alpha}-2|}}{m_{2\alpha}}$$

and

$$c_0 = \max\left\{ \inf_{r>1} c_{\alpha,r}, \inf_{r>1} c_{2\alpha,r} \right\}.$$

Suppose that either Σ is reduced and $m_\alpha \ge 2$, or Σ is not reduced and $m_{2\alpha} \ge 2$. In both cases $m_\alpha + 2m_{2\alpha} - 2 \ge 0$, and in the nonreduced case we also have $m_{2\alpha} - 2 \ge 0$. Hence $c_0 = 1 - 2/(m_\alpha + 2m_{2\alpha}) \in [0,1)$, and the minimal exponent for $\delta(m)$ in the estimates (51) is

$$\frac{1+c_0}{2} = 1 - \frac{1}{m_\alpha+2m_{2\alpha}} \in [1/2, 1).$$

Notice that $(1+c_0)/2$ can be arbitrarily close to 1. Moreover it is equal to 1/2 if and only if $m_\alpha = 2$ and $m_{2\alpha} = 0$ (i.e. in the complex rank-one case).

The wall of $A^+ \equiv (0,\infty)$ reduces in the rank-one case to the point $t = 0$. We know already from Remark 4.18 (c) that the estimate (51) is sharp for $t \to +\infty$. Therefore we only need to compare it with the asymptotic behavior for $t \to 0^+$ deduced by the explicit formula for $\Phi(m;\lambda,t)$. We have for some constant $C_m > 0$

$$\delta(m;t)^{-(1+c_0)/2} = [(2\sinh t)^{m_\alpha}(2\sinh(2t))^{m_{2\alpha}}]^{-(1+c_0)/2} \sim C_m t^{\left(\frac{m_\alpha+m_{2\alpha}}{m_\alpha+2m_{2\alpha}}\right)-(m_\alpha+m_{2\alpha})}, \qquad t \to 0^+. \tag{53}$$

The formula for $\Phi(m;\lambda,t)$ in Example 1.8 shows that the asymptotic behavior of the function $\Phi(m;\lambda,t)$ for $t \to 0^+$ can be deduced from the formulas for the analytic continuation of the hypergeometric function in a neighborhood of ∞. See [6], 2.1.4, (17) and (18), from which one concludes that for some constant $C'_{m,\lambda} > 0$

$$\frac{\Phi(m;\lambda,t)}{\Gamma(1-\lambda)} \sim C'_{m,\lambda} t^{1-(m_\alpha+m_{2\alpha})}, \qquad t \to 0^+. \tag{54}$$

Notice that $(m_\alpha + m_{2\alpha})/(m_\alpha + 2m_{2\alpha}) = 1$ when $2\alpha \notin \Sigma$. In this case, comparing (54) with (53), we conclude that the estimates (51) near $t = 0^+$ are sharp. When $2\alpha \in \Sigma$, then $(m_\alpha + m_{2\alpha})/(m_\alpha + 2m_{2\alpha}) = 1 - m_{2\alpha}/(m_\alpha + 2m_{2\alpha}) \sim 1$ when $m_\alpha \gg 2m_{2\alpha}$.

When either $0 < m_\alpha < 2$ for Σ reduced, or $0 < m_{2\alpha} < 2$ for Σ nonreduced, the above formula for c_0 becomes more complicated because at least one of the two square roots involved in the computations does not simplify. We only treat here the situations where $(m_\alpha, m_{2\alpha}) = (1, 0)$ and $(m_\alpha, m_{2\alpha}) = (2n, 1)$ ($n \in \mathbb{N}$). These multiplicities are geometric: the first case corresponds to Riemannian symmetric spaces isomorphic to the upper half-plane $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$; the second to Riemannian symmetric spaces isomorphic to the complex hyperbolic space $\mathrm{SU}(n+1,1)/\mathrm{SU}(n+1)$. Also for these multiplicity functions, the exact asymptotic behavior of $\Phi(m,\lambda,t)$ as $t \to 0^+$ can be read off from [6], 2.1.4 (18). When $(m_\alpha, m_{2\alpha}) = (1, 0)$, then

$$\Phi((1,0);\lambda,t) \sim C_\lambda \log t, \qquad t \to 0^+.$$

Since $(c_0+1)/2 \in [1/2, 1)$, the estimate (51) cannot be sharp in this case. Suppose then $(m_\alpha, m_{2\alpha}) = (2n, 1)$ with $n \in \mathbb{N}$. Then [6], 2.1.4 (18) yields (as in the general case)

$$\Phi((2n,1);\lambda,t) \sim C_{n,\lambda} t^{-2n}, \qquad t \to 0^+.$$

Since

$$c_0 = 1 - \frac{1}{n+1} \quad \text{and} \quad \frac{c_0+1}{2} = 1 - \frac{1}{2(n+1)}$$

for all $n \in \mathbb{N}$, the error between exponent of the power of t in the estimate (51) near $t = 0^+$ and the correct asymptotic behavior of $\Phi((2n,1);\lambda,t)$ as $t \to 0^+$ is $1 - (2n+1)/(2n+2) = 1/(2n+2)$, which tends to 0 as $n \to +\infty$.

5. Estimates for the Θ-hypergeometric functions

The estimates for the Harish-Chandra series established in Section 4 can be employed for estimating the Θ-hypergeometric functions by means of their defining formula (23).

For $R > 0$, let $\mathfrak{a}^*_\mathbb{C}(R)$ be the domain in $\mathfrak{a}^*_\mathbb{C}$ defined in (29). Let $\Theta \subset \Pi$ be fixed and define $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ as the set of all $\lambda \in \mathfrak{a}^*_\mathbb{C}$ satisfying

$$\begin{aligned} |\mathrm{Re}\langle\lambda,\alpha\rangle| < R \quad &\text{for all } \alpha \in \langle\Theta\rangle^+, \\ \mathrm{Re}\langle\lambda,\alpha\rangle < R \quad &\text{for all } \alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+. \end{aligned}$$

Then $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ is the largest W_Θ-invariant open subset of $\mathfrak{a}^*_\mathbb{C}(R)$. Every $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ has the property that $w\lambda \in \mathfrak{a}^*_\mathbb{C}(R)$ for all $w \in W_\Theta$. Hence we can apply the estimates of the previous section to all the $\Phi(m; w\lambda, a)$ occurring at the right-hand side of (23). Moreover, $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ intersects only finitely many singular hyperplanes of the function c^+_Θ and of the numerator function n^-_Θ. These singularities can therefore be removed by multiplying by a suitable polynomial in λ (which will also depend on Θ and R). See Corollaries 5.2 and 5.4 below. The resulting estimates for the Θ-hypergeometric functions, initially determined only for $a \in A^+$ and for generic $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$, will then be extended by continuity and W_Θ-invariance to the entire $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ and A_Θ. When R ranges among the positive real numbers, the domains $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ cover $\mathfrak{a}^*_\mathbb{C}$. Thus our estimates will be uniform in $a \in A_\Theta$ and locally uniform in $\lambda \in \mathfrak{a}^*_\mathbb{C}$.

Lemma 5.1. *Let $R > 0$, $r \in \mathbb{R}$ and $\alpha \in \Sigma$. Suppose the set $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ intersects the complex hyperplane $\mathcal{H}_{r,\alpha} = \{\lambda \in \mathfrak{a}^*_\mathbb{C} : \lambda_\alpha = r\}$.*

(a) *If $\alpha \in \langle\Theta\rangle$, then $|r| < R/\langle\alpha,\alpha\rangle$.*
(b) *If $\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$, then $r < R/\langle\alpha,\alpha\rangle$.*

□

The next corollary is a straightforward consequence of Lemmas 2.7 and 5.1 and of the classical asymptotics of ratios of gamma functions: For arbitrarily fixed $a, b \in \mathbb{C}$ and any small $\varepsilon > 0$ we have

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left[1 + O(z^{-1})\right] \tag{55}$$

for all $z \in \mathbb{C}$ with $|\arg z| \leq \pi - \varepsilon < \pi$ (see e.g. [6], 1.18(4)). For $R > 0$ and $\alpha \in \Sigma$ we shall employ the notation

$$R_\alpha := \left[\frac{R}{\langle \alpha, \alpha \rangle}\right],$$

where $[x]$ denotes the integer part of $x \in [0, +\infty)$.

Corollary 5.2. *Suppose $R > 0$. Then the set $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ intersects only finitely many singular hyperplanes of the function $c^+_\Theta(m; w\lambda)$ for all $w \in W_\Theta$. More precisely, set*

$$\pi_{\Theta,R}(\lambda) := \prod_{\alpha \in \langle \Theta \rangle_{\mathrm{i}}} \prod_{n=0}^{R_\alpha} (\lambda_\alpha + n). \tag{56}$$

Then $\pi_{\Theta,R}$ is a W_Θ-invariant polynomial function of $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ with the property that $\pi_{\Theta,R}(\lambda) c^+_\Theta(m; w\lambda)$ is holomorphic in $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ for all $w \in W_\Theta$. Moreover, there are positive constants $C_{\Theta,R,m}$ and $r(\Theta, R, m)$ such that

$$\left|\pi_{\Theta,R}(\lambda) c^+_\Theta(m; w\lambda)\right| \leq C_{\Theta,R,m}(1 + |\lambda|)^{r(\Theta,R,m)}$$

for all $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ and all $w \in W_\Theta$. □

Let $R > 0$ be fixed. By definition, the singularities of c^-_Θ in $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ are (counted with multiplicities) among those of n^-_Θ. By Theorem 3.5, this holds also for the Θ-hypergeometric functions $\varphi_\Theta(m; \lambda, a)$. The singularities of n^-_Θ listed in Lemma 2.7 together with Lemma 5.1 yield the following corollary.

Corollary 5.3. *Suppose $R > 0$. Then $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ intersects only finitely many singular hyperplanes of the numerator function $n^-_\Theta(m; \lambda)$. More precisely, let $\alpha \in \Sigma^+_{\mathrm{i}} \setminus \langle \Theta \rangle^+_{\mathrm{i}}$. Then $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ intersects:*

(a) *no hyperplanes $\mathcal{H}_{-\frac{m_\alpha}{2}+(2n-1),\alpha}$ ($n \in \mathbb{N}$), if $R \leq \langle \alpha, \alpha \rangle (1 - m_\alpha/2)$ (which is possible only when $m_\alpha < 2$);*

(b) *at most the hyperplanes $\mathcal{H}_{-\frac{m_\alpha}{2}+(2n-1),\alpha}$ with $n = 1, 2, \ldots, R_{\alpha,1}$, if $R > \langle \alpha, \alpha \rangle (1 - m_\alpha/2)$, where $R_{\alpha,1} := \left[\frac{1}{2}\left(\frac{m_\alpha}{2} + \frac{R}{\langle \alpha,\alpha \rangle} + 1\right)\right]$;*

(c) *no hyperplanes $\mathcal{H}_{-\frac{m_\alpha}{2}-m_{2\alpha}+2n,\alpha}$ ($n \in \mathbb{N}$), if $R \leq \langle \alpha, \alpha \rangle (2 - m_\alpha/2 - m_{2\alpha})$ (which is possible only when $m_\alpha/2 + m_{2\alpha} < 2$);*

(d) *at most the hyperplanes $\mathcal{H}_{-\frac{m_\alpha}{2}-m_{2\alpha}+2n,\alpha}$ with $n = 1, 2, \ldots, R_{\alpha,2}$, if $R > \langle \alpha, \alpha \rangle (2 - m_\alpha/2 - m_{2\alpha})$, where $R_{\alpha,2} := \left[\frac{1}{2}\left(\frac{m_\alpha}{2} + m_{2\alpha} + \frac{R}{\langle \alpha,\alpha \rangle}\right)\right]$.*

□

We set $R_{\alpha,1} := 0$ in case (a) of Corollary 5.3, and $R_{\alpha,2} := 0$ in case (c). With the given $R_{\alpha,1}$ and $R_{\alpha,2}$ for the cases (b) and (d) and with the usual convention that empty products are equal to 1, we define

$$n^-_{\Theta,R}(m; \lambda) := \prod_{\alpha \in \Sigma^+_{\mathrm{i}} \setminus \langle \Theta \rangle^+_{\mathrm{i}}} \prod_{n_1=1}^{R_{\alpha,1}} \left(-\frac{\lambda_\alpha}{2} - \frac{m_\alpha}{4} - \frac{1}{2} + n_1\right) \prod_{n_2=1}^{R_{\alpha,2}} \left(-\frac{\lambda_\alpha}{2} - \frac{m_\alpha}{4} - \frac{m_{2\alpha}}{2} + n_2\right). \tag{57}$$

Corollary 5.4. *Suppose $R > 0$ is fixed. Then $n^-_{\Theta,R}(m; \lambda)$ is a W_Θ-invariant polynomial function in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ with the property that*

$$n^-_{\Theta,R}(m; \lambda)\varphi_\Theta(m; \lambda, a)$$

is holomorphic in $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ *for all* $a \in A_\Theta$. *Moreover, there are positive constants* $C'_{\Theta,R,m}$ *and* $r'(\Theta,R,m)$ *such that*

$$\left|n^-_{\Theta,R}(m;\lambda)c^-_\Theta(m;\lambda)\right| \leq C'_{\Theta,R,m}(1+|\lambda|)^{r'(\Theta,R,m)}$$

for all $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$.

Proof. The function $n^-_{\Theta,R}(m;\lambda)\varphi_\Theta(m;\lambda,a)$ is holomorphic in $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ when so is the function $n^-_{\Theta,R}(m;\lambda)n^-_\Theta(m;\lambda)$. Using (21) and the functional relation $z\Gamma(z)=\Gamma(z+1)$, one easily verifies the equality

$$n^-_{\Theta,R}(m;\lambda)n^-_\Theta(m;\lambda) = \prod_{\alpha\in\Sigma^+_{\mathrm{i}}\setminus\langle\Theta\rangle^+_{\mathrm{i}}} \Gamma\left(-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}+\frac{1}{2}+R_{\alpha,1}\right)\Gamma\left(-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{4}-\frac{m_{2\alpha}}{2}+1+R_{\alpha,2}\right). \tag{58}$$

The conditions defining $R_{\alpha,1}$ and $R_{\alpha,2}$ ensure then that the product (58) is nonsingular in $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$.
The final estimate is a straightforward consequence of (55). □

Theorem 4.17 proves estimates for the Harish-Chandra series $\Phi(m;\lambda,a)$ for all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}(R)$ and $a \in A^+$: multiplication by the polynomial $p_R(\lambda)$ of (40) cancels all λ-singularities of $\Phi(m;\lambda,a)$ in $\mathfrak{a}^*_{\mathbb{C}}(R)$; multiplication by $\delta(m;a)^{(c+1)/2}$, for a suitable $c\in[0,1)$, cancels the singularities along the walls of A^+. We make $p_R(\lambda)$ invariant under W_Θ by defining

$$p_{\Theta,R}(\lambda) := \prod_{w\in W_\Theta} p_R(w\lambda). \tag{59}$$

Then $p_{\Theta,R}(\lambda)$ is a polynomial in λ of degree $\deg p_{\Theta,R} = |W_\Theta|\deg p_R$. Moreover, $p_{\Theta,R}\equiv 1$ whenever so is p_R. By restricting the variable λ to the W_Θ-invariant domain $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)\subset\mathfrak{a}^*_{\mathbb{C}}(R)$, we can extend Theorem 4.17 to $\Phi(m;w\lambda,a)$ for all $w\in W_\Theta$ and obtain for all $a\in A^+$ and $\lambda\in\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$

$$\left|p_{\Theta,R}(\lambda)\delta(m;a)^{(c+1)/2}\Phi(m;w\lambda,a)\right| \leq C_{R,c,m}(1+|\lambda|)^{|W_\Theta|\deg p_R}e^{(c\rho(m)+\operatorname{Re} w\lambda)(\log a)}. \tag{60}$$

The right-hand side of (60) can be made W_Θ-invariant by replacing $\operatorname{Re} w\lambda(\log a)$ with

$$\operatorname{Re} W_\Theta\lambda(\log a) := \max_{w\in W_\Theta} \operatorname{Re} w\lambda(\log a)$$

and $a\in A^+$ with the element a_Θ of the W_Θ-orbit of a lying in $\overline{A^+}$, that is

$$\{a_\Theta\} := W_\Theta a \cap \overline{A^+}. \tag{61}$$

Notice that $\operatorname{Re} W_\Theta\lambda(\log a) = \operatorname{Re} W_\Theta\lambda(\log a_\Theta)$ for all $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$ and $\alpha\in A_\Theta$. Combining all estimates proven in this section, we reach the following estimate for the Θ-hypergeometric functions.

Theorem 5.5. *Suppose* $m\in\mathcal{M}^+$ *and* $R>0$. *Let* $\pi_{\Theta,R}(\lambda)$, $p_{\Theta,R}(\lambda)$, $n^-_{\Theta,R}(m;\lambda)$ *be the polynomials defined in (56), (59) and (57), respectively. Let* $\delta(m;a)$ *be defined by (2). Finally, let* $c\in[0,1)$ *be the constant (depending only on the multiplicity function* m*) which can be computed from Lemma 4.11. Then there are constants* $C_{\Theta,R,c,m}>0$ *and* $d(\Theta,R,m)\geq 0$ *such that*

$$\left|\pi_{\Theta,R}(\lambda)p_{\Theta,R}(\lambda)n^-_{\Theta,R}(m;\lambda)\delta(m;a)^{(c+1)/2}\varphi_\Theta(m;\lambda,a)\right| \leq C_{\Theta,R,c,m}(1+|\lambda|)^{d(\Theta,R,m)}e^{(c\rho(m)+\operatorname{Re} W_\Theta\lambda)(\log a_\Theta)} \tag{62}$$

for all $a\in A_\Theta$ *and* $\lambda\in\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$.

We can choose $p_{\Theta,R}\equiv 1$ *for all sufficiently small* $R>0$ *(e.g.* $R\leq 1/(2Q^2)$ *with* Q *as in (38)). Moreover, we have* $n^-_{\Theta,R}\equiv 1$ *for all* R *when* $\Theta=\Pi$, *and* $\pi_{\Theta,R}\equiv 1$ *for all* R *when* $\Theta=\emptyset$.

Proof. For all $a\in A^+$ and for generic $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$ the Θ-hypergeometric function φ_Θ is defined by (23), which implies the equality

$$\begin{aligned}&\pi_{\Theta,R}(\lambda)p_{\Theta,R}(\lambda)n^-_{\Theta,R}(m;\lambda)\delta(m;a)^{(c+1)/2}\varphi_\Theta(m;\lambda,a)\\&\quad=\left[n^-_{\Theta,R}(m;\lambda)c^-_\Theta(m;\lambda)\right]\sum_{w\in W_\Theta}\left[\pi_{\Theta,R}(\lambda)c^+_\Theta(m;w\lambda)\right]\left[p_{\Theta,R}(\lambda)\delta(m;a)^{(c+1)/2}\Phi(m;w\lambda,a)\right]\end{aligned}$$

Corollaries 5.4 and 5.2 together with (60) prove the required estimate (62) for generic $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ and $a \in A^+$. The left-hand side of (62) is a holomorphic function of $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ by Corollary 5.4. Moreover, it is continuous and W_Θ-invariant in $a \in A_\Theta$. The estimate therefore extends by continuity for all $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ and $a \in A_\Theta$. □

From Theorem 5.5 we deduce new estimates for the spherical functions on a NCC symmetric space G/H. We assume the notation of Chapter 8 in [20]. In particular, we denote by Σ_0 the set of compact roots, with Weyl group W_0, and by Σ_0^+ a suitable choice of positive compact roots. Let $c^0_{\max} = (W_0 \cdot \overline{\mathfrak{a}^+})^0$ be the interior of the maximal cone associated with the causal structure, and let $S^0 := H\exp(c^0_{\max})H$. A spherical function $\varphi : S^0/H \to \mathbb{C}$ is uniquely determined by its W_0-invariant restriction to $\exp(c^0_{\max})$. We denote by a_0 the unique element of the W_0-orbit of $a \in \exp(c^0_{\max})$ lying in $\overline{A^+}$ and set $\operatorname{Re} W_0\lambda(\log a) := \max_{w\in W_0} \operatorname{Re} w\lambda(\log a)$. As common in this context, we write $\delta(a)$ instead of $\delta(m;a)$ and ρ instead of $\rho(m)$.

Corollary 5.6. *Let $\varphi_\lambda : S^0/H \to \mathbb{C}$ denote the spherical function of spectral parameter $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ on the NCC symmetric space G/H. Let $c \in [0,1)$ be the constant (depending only on the multiplicities of G/H) from Lemma 4.11. Suppose $R > 0$ and define $\mathfrak{a}^*_{\mathbb{C},0}(R)$ as the set of $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ with $|\operatorname{Re}\langle\lambda,\alpha\rangle| < R$ for all $\alpha \in \Sigma_0^+$ and $\operatorname{Re}\langle\lambda,\alpha\rangle < R$ for all $\alpha \in \Sigma^+ \setminus \Sigma_0^+$. Then there is a polynomial $q_R(\lambda)$ and constants $C_{R,c} > 0$ and $d(R) > 0$ such that*

$$\left| q_R(\lambda)\delta(a)^{(c+1)/2}\varphi_\lambda(a)\right| \le C_{R,c}(1+|\lambda|)^{d(R)} e^{(c\rho+\operatorname{Re} W_0\lambda)(\log a_0)}$$

for all $a \in \exp(c^0_{\max})$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C},0}(R)$. □

The polynomials $\pi_{\Theta,R}(\lambda)$and $p_{\Theta,R}(\lambda)$ appearing on the left-hand side of (62) are in fact unnecessary. They would allow $n^-_{\Theta,R}(m;\lambda)\varphi_\Theta(m;\lambda,a)$ to have singularities in $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$, but we know from Corollary 5.4 that this function is holomorphic in $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ for all $a \in A_\Theta$. It is possible to apply Corollary 4.3 and improve the estimates in λ by removing these superfluous polynomials, but in doing so we somehow worsen the estimates in the a-variable for large a.

Corollary 5.7. *Keep the notation of Theorem 5.5. Then there are constants $\varepsilon_R > 0$ and $d'(\Theta,R,m) \ge 0$ with the following property: For every $\varepsilon \in (0,\varepsilon_R]$ there is a constant $C_{\Theta,R,c,m,\varepsilon} > 0$ such that*

$$\left| n^-_{\Theta,R}(m;\lambda)\delta(m;a)^{(c+1)/2}\varphi_\Theta(m;\lambda,a)\right| \le C_{\Theta,R,c,m,\varepsilon}(1+|\lambda|)^{d'(\Theta,R,m)} e^{(c\rho(m)+\operatorname{Re} W_\Theta\lambda)(\log a_\Theta)+\varepsilon|\log a|} \tag{63}$$

for all $a \in A_\Theta$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(R)$.

Proof. Let $\varepsilon_R > 0$ be chosen so that the ε_R-neighborhood $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)_{\varepsilon_R}$ of $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ is contained in $\mathfrak{a}^*_{\mathbb{C},\Theta}(2R)$. Then $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)_\varepsilon \subset \mathfrak{a}^*_{\mathbb{C},\Theta}(2R)$ for all $\varepsilon \in (0,\varepsilon_R]$. The estimate of Theorem 5.5 in the larger domain $\mathfrak{a}^*_{\mathbb{C},\Theta}(2R)$ gives for all $a \in A_\Theta$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C},\Theta}(2R)$

$$\left| q_{\Theta,R}(\lambda)n^-_{\Theta,2R}(m;\lambda)\delta(m;a)^{(c+1)/2}\varphi_\Theta(m;\lambda,a)\right| \le C_{\Theta,2R,c,m}(1+|\lambda|)^{d(\Theta,2R,m)} e^{(c\rho(m)+\operatorname{Re} W_\Theta\lambda)(\log a_\Theta)}$$

where

$$q_{\Theta,R}(\lambda) := \pi_{\Theta,2R}(\lambda)p_{\Theta,2R}(\lambda)\frac{n^-_{\Theta,2R}(m;\lambda)}{n^-_{\Theta,R}(m;\lambda)}$$

is a polynomial. For fixed $a \in A_\Theta$, consider the elements $w\log a_\Theta$ for $w \in W_\Theta$. They satisfy $|w\log a_\Theta| = |\log a|$ for all w. The required inequality is obtained by applying Corollary 4.3 to the holomorphic function $H(\lambda) := q_{\Theta,R}(\lambda)n^-_{\Theta,2R}(m;\lambda)\delta(m;a)^{(c+1)/2}\varphi_\Theta(m;\lambda,a)$, the polynomial $q_{\Theta,R}(\lambda)$ and the elements $w\log a_\Theta$. □

Remark 5.8. (a) When $\Theta \neq \emptyset$, the estimates of Theorem 5.5 and Corollary 5.7 are not sharp in many senses. In fact, they are even getting worse when the Θ-hypergeometric functions are becoming more regular. This can be observed at three points. First, the domain $\mathfrak{a}^*_{\mathbb{C},\Theta}(R)$ for the variable λ on which the estimates hold is getting smaller when Θ increases. In particular, it is always a compact subset of $\mathfrak{a}^*_{\mathbb{C}}$ in the most regular case $\Theta = \Pi$. Second, the term $\delta(m;a)^{(c+1)/2}$ is independent of Θ. It is necessary for the case $\Theta = \emptyset$, but completely superfluous in the case $\Theta = \Pi$. The Θ-hypergeometric function φ_Θ is regular on A_Θ. This suggests that the correct factor should be

$$\delta_\Theta(m) := \prod_{\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+} \left|e^\alpha - e^{-\alpha}\right|^{m_\alpha}.$$

Third, the polynomial factor in λ on the right-hand side of the estimates is unnecessary for $\Theta = \Pi$ (see below). However, being of polynomial nature, this additional factor gives no problems in most of the applications. The estimates of Theorem 5.5 and Corollary 5.7 are nevertheless the first piece of information about the behavior of the Θ-hypergeometric functions for all values of the spectral parameter along the walls of A_Θ, also for the geometric case of spherical functions on NCC spaces. The behavior predicted along the walls will be good enough to apply the Θ-hypergeometric Fourier transform to functions without compact support in A_Θ.

(b) In Proposition 6.1(1) and Theorem 3.15 of [33], Opdam has proven estimates for the hypergeometric functions associated with root systems. We state them for the functions φ_Π, which differ from the hypergeometric functions for root system only by a constant factor: *Suppose $m \in \mathcal{M}^+$ is fixed. Then*

$$|\varphi_\Pi(m;\lambda,a)| \leq |W|^{1/2}\, c_\Pi^+(m;\rho(m))\, e^{\max_{w\in W} \mathrm{Re}(w\lambda(\log a))} \tag{64}$$

for all $a \in A$.

Opdam could even extend the estimates (in a slightly more complicated version) to a tubular neighborhood of A in $A_\mathbb{C}$. See [33], Proposition 6.1(2).

The proving techniques of Opdam's estimates are very interesting: he deduces the results for the hypergeometric functions from the estimates for another family of eigenfunctions of the Dunkl-Cherednik operators, which he calls the *non-symmetric hypergeometric functions*. The Dunkl-Cherednik operators are differential-reflection operators of *first* order, and this allows Opdam to apply a clever trick to get the required estimates in the non-symmetric case. The hypergeometric functions are obtained from the non-symmetric hypergeometric functions by averaging over W, and this easily yields the estimates reported in (64). We refer to Section 3 in [33] and Section 7 of [34] for further details.

The geometric interpretation of Opdam's non-symmetric hypergeometric functions in the Riemannian case is to our knowledge still an open problem. The construction of the non-symmetric hypergeometric functions in the context of Θ-hypergeometric functions is a challenging project with interests also for the geometric case of NCC symmetric spaces.

(c) In the geometric context of Riemannian symmetric spaces, the so-called Harish-Chandra estimates provide sharp estimates for the spherical functions. Their proof employs *both* the integral formulas and the system of differential equations. The integral formulas provide shift inequalities which reduce to the estimation of the spherical functions for the spectral parameter $\lambda = 0$. Similar techniques of the Riemannian case cannot be applied in the context of hypergeometric functions associated with root systems (and therefore in the general context of Θ-hypergeometric functions) because of the absence of integral formulas. In fact, they cannot be applied even in the geometric setting of the NCC symmetric spaces, where explicit integral formulas are given only on a certain subset of $\mathfrak{a}_\mathbb{C}^*$. For arbitrary values of the spectral parameter λ (among them for $\lambda = 0$), these integral formulas have been recently extended in [28] by the method of meromorphic continuation via Bernstein-Sato polynomials. They are however not completely explicit due to insufficient information on the Bernstein-Sato polynomials and their corresponding differential operators. The estimates for the spherical functions deduced from them in [2] are consequently not sharp. For instance they cannot predict any behavior of the spherical functions along the walls of $\exp(c_{\max})$.

Theorem 5.5 implies that for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ the function

$$\frac{\varphi_\Theta(m;\lambda,a)}{n_\Theta^-(m;\lambda)}\,\delta(m;a)$$

vanishes along the walls of A_Θ. This property ensures that the integral defined the Θ-hypergeometric transform (3) converges outside of the λ-singular set of φ_Θ also when applied to functions without compact support in A_Θ. Indeed, since $(1-c)/2 > 0$, we can even admit some singularity for f along the walls of A_Θ. We have for instance the following proposition, which applies in particular to the spherical transform on NCC symmetric spaces.

Proposition 5.9. *Let $m \in \mathcal{M}^+$ and $\Theta \subset \Pi$ be fixed. Suppose f is a W_Θ-invariant function such that $\delta(m)^{(1-c)/2} f$ is continuous and compactly supported in $\overline{A_\Theta}$. Then*

(a) *The Θ-hypergeometric Fourier transform $\mathcal{F}_\Theta f(m;\lambda)$ is a W_Θ-invariant meromorphic function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Its singularities are located (counting multiplicities) in the polar set of the numerator function $n_\Theta^-(m;\lambda)$. When Σ is reduced, then all singularities are simple poles.*

(b) *Suppose Σ is reduced and m is an even multiplicity function. Let $e_\Theta^-(m;\lambda)$ be the polynomial function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ defined in (25). Then $e_\Theta^-(m;\lambda)\mathcal{F}_\Theta f(m;\lambda)$ is a W_Θ-invariant entire function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.* □

Remark 5.10. Since for $A^+ \ni a \to +\infty$

$$\delta(m;a)^{(c+1)/2} \sim e^{(c+1)\rho(m)(\log a)},$$

we obtain, for almost all λ and for a certain constant $C_{R,\lambda,m} > 0$,

$$\varphi_\Theta(m;\lambda,a) \lesssim C_{R,\lambda,m} e^{(\rho(m)+W_\Theta\lambda)(\log a)}$$

whenever $\lambda \in \mathfrak{a}_{\mathbb{C},\Theta}^*(R)$ and $A^+ \ni a \to +\infty$. This suggests the possibility for the Θ-hypergeometric Fourier transform of a Schwartz space theory in which, as in the Riemannian case, the correct definition of rapid decrease of a function f requires that for every $m \in \mathbb{N}$

$$f(a) = O\left(e^{-\rho(m)(\log a)}(1+|\log a|)^m\right)$$

for $a \in A^+$ (see e.g. [9], §6.1).

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